Maximum-Likelihood Method for Image Reconstruction of Gamma-Ray-Detectors

Seminar of Statistical Methods

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Overview

Maximum Likelihood Estimation

Estimator:

A function of random variables to estimate unknown parameters of a theoretical probability distribution.

Example: Arithmetic mean

$$\widehat{x} = \frac{x_1 + ... + x_n}{n} \tag{1}$$

for $x_1 + ... + x_n$ independent, identically distributed random variables

Two types of estimators: **point estimator** and **interval estimator**

Point estimator:

- Every sample can be assigned one value:

$$T : (X, \mathcal{A}) \rightarrow (E, \mathcal{E}) \qquad (2)$$

  with $(X, \mathcal{A})$ a statistic model and $(E, \mathcal{E})$ the measurable space. For every $e \in E$ is there an $e \in \mathcal{E}$. For all $M \in \mathcal{E}$ is $T^{-1} \in \mathcal{A}$.
- Point estimators are used to approximate values for unknown physical variables

Interval estimator:

- For every $e \in E$ there is a set $\mathcal{A}(e)\ \{x \in X | m \in T(x)\} \in \mathcal{E}$
- statistical estimator represented by a set of points in a parameters space
- For a single parameters a to be estimated, the interval estimator gives a certain confidence interval

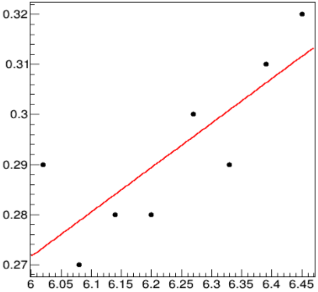
Example: Toss of coins:
53 tosses of coins with random variable $X = (X_1, ..., X_{53})$ ($X_i = 1$ if coin shows head) and $P(X_i = 1) = p$.

- from data derived value $\widehat{p}$ is point estimator
- from data derived interval $[a, b]$ for p is interval estimator

Method of Least Squares

= estimating parameters by **minimizing the squared discrepancies** between **observed** and **expected** data.

- function $Y = f(X)$ + noise
- $\widehat{\beta}$: value that minimizes $\sum_{i=1}^{n}(y_i - f_b(x_i))^2$



Method of Moments

Basic idea:

- first sample moment M_1 to the first theoretical moment $E(X)$
- continue to equate sample moments and theoretical moments for $k = 1, 2, ..$ as number of parameters
- solve equations for the parameters

Definitions:
Moment: $m_k := E(X^k)$

Sample moment:
$M_k := 1/n \sum_{i=1}^{n} X_i^k$

Example:
Bernoulli random variables $X_1, X_2, ..., X_n$ with parameter μ?

- first theoretical moment is: $E(X_i) = \mu$
- First corresponding sample moment: $E(X) = \mu = 1/n \sum_{i=1}^{n} X_i$
- Equation: $\widehat{\mu} = 1/n \sum_{i=1}^{n} X_i$

Likelihood estimation:

Idea: good estimation for unknown parameter θ is the value of θ that maximizes the **likelihood** of getting the observed data.

Likelihood function:
random sample $X_1, X_2, ..., X_n$ depending one unknown parameters $\theta_1, \theta_2, ..., \theta_m$ with probability density function of each X_i is $f(x_i; \theta_1, \theta_2, ..., \theta_m)$

The joint probability density function (Likelihood function) is:

$$L(\theta_1, \theta_2, ..., \theta_m) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2, ..., \theta_m) \tag{3}$$

Maximum Likelihood estimator: $\widehat{\theta}_i = u_i(X_1, X_2, ..., X_n)$

for $[u_1(x_1, x_2, ..., x_n), u_2(x_1, x_2, ..., x_n), ..., u_m(x_1, x_2, ..., x_n)]$ the m-tuple that maximizes the likelihood function.

These values $u_1, u_2, ..., u_n$ are the **maximum likelihood estimates** for θ_i for $i = 1, 2, ..., m$

Time for an example ...

Example:

10 randomly selected cats from a *colonia felina* have the weights:

2.9, 3.4, 4.0, 3.7, 4.2, 2.8, 3.9, 3.8, 4.5, 3.5

What is the maximum likelihood estimator of μ?

Probability density function of X_i is Gaussian, therefore the Likelihood function is:

$$L(\mu, \sigma) = \sigma^{-n}(2\pi)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \tag{4}$$

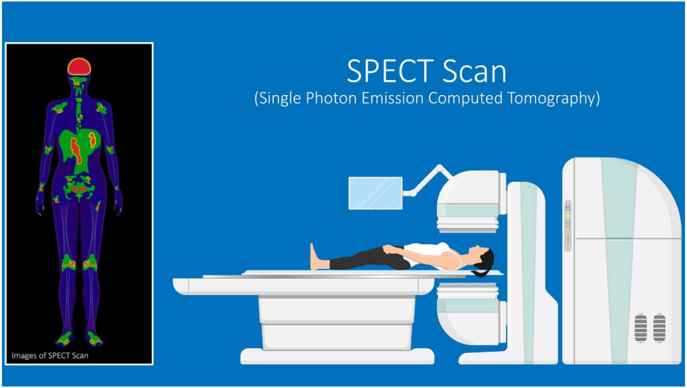
To find the maximum, we use **log of the likelihood function**:

$$\log L(\mu, \sigma) = -n\log(\sigma) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 \tag{5}$$

The resulting estimator is:

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X} = 3.67kg \tag{6}$$

Image Reconstruction of Gamma-Ray Detectors



https://healthlibrary.askapollo.com/what-is-a-spect-scan-commonly-used-for/

SPECT: nuclear imaging technique used in clinical environment. Provides high efficiency in diagnosing for example Alzheimer's and Parkinson.

Setup of gamma camera

Main components:

- lead collimator
- a large scintillating crystal
- 50-100 photomultiplier tubes (PMT)
- shielding of lead around the camera

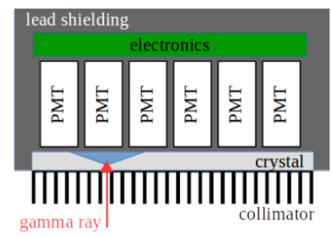


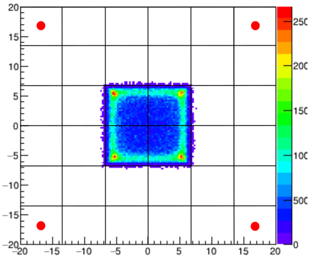
Image reconstruction

Centroid method

- Coordinates of an event j are reconstructed by means of the centroid of the signal q_{ij} of all M pixels (all M PMTs)

$$X_j = \frac{\sum_i^M x_i \cdot q_{ji}}{\sum_i^M q_{ji}} \qquad Y_j = \frac{\sum_i^M y_i \cdot q_{ji}}{\sum_i^M q_{ji}}$$

- Simple, robust method, but not very accurate



Flood field image from simulations reconstructed with centroid

Statistical method

= find the best match between observed and expected detector signals

Advantages:

- random nature of gamma detection is taken into account
- potentially smaller distortions
- larger useful field of view
- better filtering of noise events

Challenge: Need of detailed knowledge of the average signal amplitudes as function of the event position of each pixel → **Light Reponse Function** of each pixel (LRF)

Light Reponse Function (LRF)

- The number of photons generated in a scintillation process is Q. The probability for a $i-$th pixel to collect q_i photons is **Poisson distributed**:

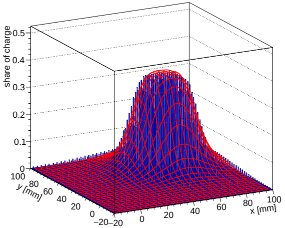
$$P_i(q_i) = \frac{\mu_i^{q_i} \exp^{-\mu_i}}{q_i!} \quad (7)$$

- Expectation for number of photons detected by the $i-$th pixel with f_i LRF:

$$\mu_i = Q \cdot f_i(\overrightarrow{r}) \quad (8)$$

Total number of photons Q for event j for all M pixels:

$$Q_j = \sum_i^M q_{ij} \quad \rightarrow \quad f_{ij} = q_{ij}/Q_j \quad (9)$$



= find set of coordinates $\widehat{X}$ and $\widehat{Y}$ and energy $\widehat{Q}$ to maximize likelihood of getting the result of the measurement.

Likelihood function

- Likelihood function of Poisson distribution for M pixel:

$$L = \prod_{i}^{M} P(q_i|\mu_i) \tag{10}$$

- Simplified as:

$$ln(L) = \sum_{i}^{M} P(q_i|\mu_i) = \sum_{i}^{M}(q_i \ln(\mu_i) - \mu_i) - \sum_{i}^{M}(q_i!) \tag{11}$$

- For the reconstruction:

$$ln(L(\widehat{r}, \widehat{Q})) = \sum_{i}^{M} (q_i \ln(\widehat{Q} f_i(\widehat{r})) - \widehat{Q} f_i(\widehat{r}) + const \tag{12}$$

- $\widehat{Q}$ can be expressed as:

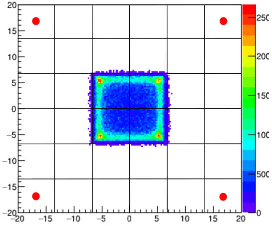
$$\widehat{Q}(\vec{r}) = \frac{\sum_i^M q_i}{\sum_i^M f_i(\vec{r})} \tag{13}$$

$$\boxed{-\ln(L(\vec{r})) = -\sum_{i}^{N} \left(q_i \cdot \ln \left(f_i(\vec{r}) \cdot \frac{\sum_i^N q_i}{\sum_i^N f_i(\vec{r})} \right) - f_i(\vec{r}) \cdot \frac{\sum_i^N q_i}{\sum_i^N f_i(\vec{r})} \right)}$$

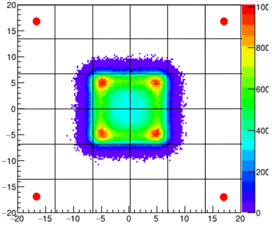
$$-\ln(L(\vec{r})) = -\sum_{i}^{N}\left(q_i \cdot \ln\left(f_i(\vec{r}) \cdot \frac{\sum_i^N q_i}{\sum_i^N f_i(\vec{r})}\right) - f_i(\vec{r}) \cdot \frac{\sum_i^N q_i}{\sum_i^N f_i(\vec{r})}\right)$$

Implementation:

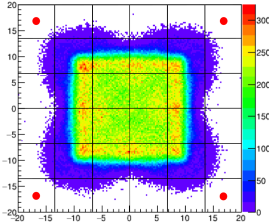
- The reconstruction was implemented in `root`.
- Negative Log-Likelihood function is minimized using `Minuit2 Minimizer root`.
- If LRFs known from calibration grid, event position can directly reconstructed
- If LRF not known, iterative technique is applied (reconstruction of flood field image + Gaussian noise after every iteration)
- Decide which shape LRF has: Gaussian or other bell-shaped function



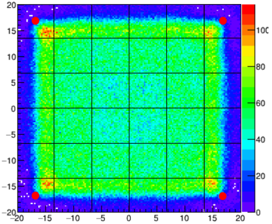
Centroid Method



with LRF after first iteration

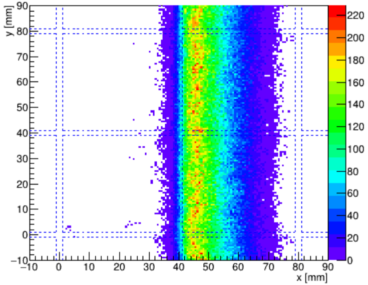


with LRF after 3 iterations

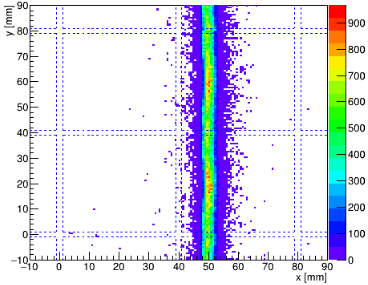


with LRF after 17 iteration

Centroid vs. Maximum Likelihood



Centroid Method after correction



Maximum Likelihood Method after correction

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Examples for point estimators:

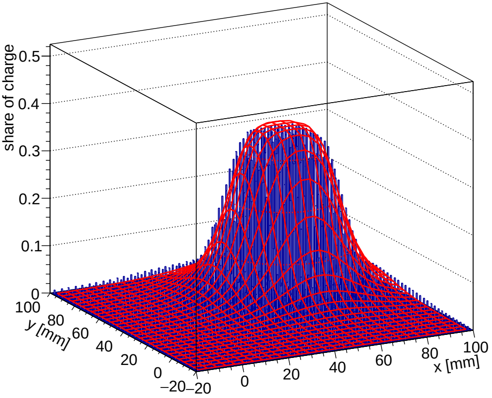
- Method of Least Squares
- Method of Moments
- Maximum Likelihood Estimation

Iterative method:

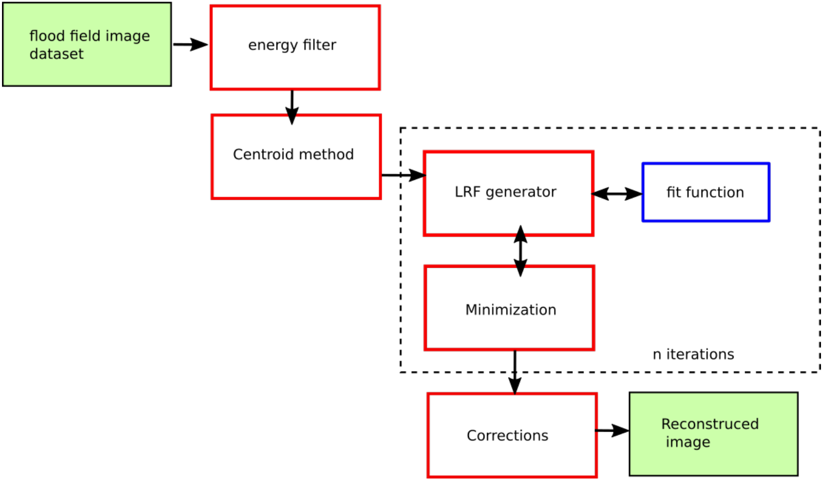
- Start value for LRF (position reconstructed with Centroid)
- Reconstruction of flood field image
- From flood field image: new LRF

Grid method:

- Calibration grid of point over field of view
- From distribution of charge over pixel for each point LRF is estimated



LRF for one pixel after interpolation


flood field image dataset
energy filter
Centroid method
LRF generator
fit function
Minimization
n iterations
Corrections
Reconstruced image