Autocovariance of wide sense stationary random processes

A tool for characterizing cross-talking effects in pixelated X-ray photon counting detectors

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Outline

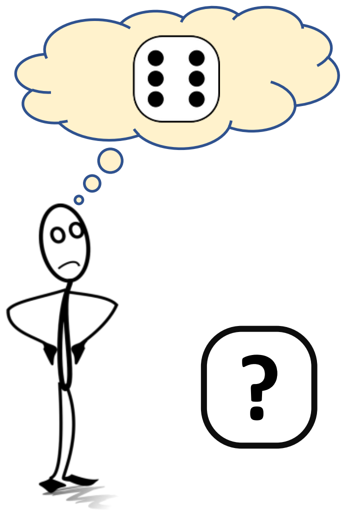
- Theoretical background
- Random events for X-ray photon counting detectors (PCD)
- Autocorrelation and Autocovariance interpretation
- Simulation study
- Experimental case

Theoretical background

Random variable

- *A random variable is the output of a random event, e.g. toss a coin, dice roll etc.*

The exact outcome of a random event is unpredictable



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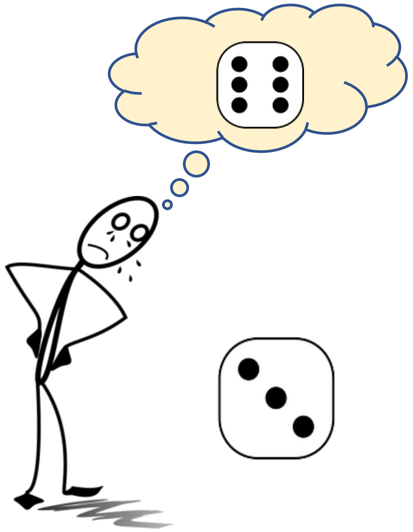


Probability distribution

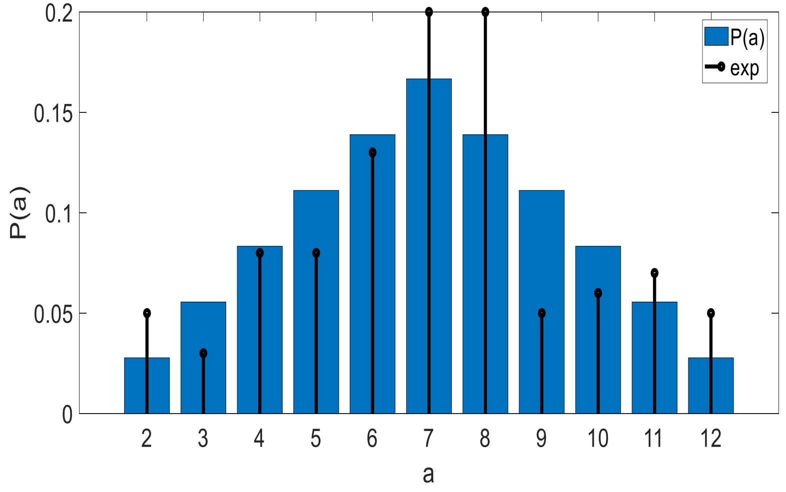
The exact outcome of a random event is unpredictable, however it is possible to determine its statistical properties!

- If a random event ‘a’ is repeated many times, it will produce a distribution of outcomes
- This distribution can be mathematically described by a probability distribution p(a)

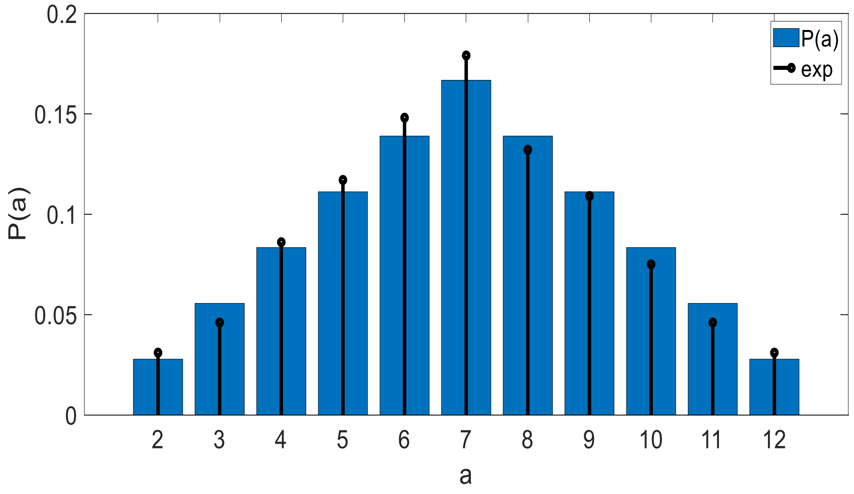
Example: outcome of the sum of two dice

a	P(a)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

N=100 random events



N=1000 random events



Expected value

- For a probability distribution, the Expected value (mean) is defined as follow:

$$\mu = E\{a\} = \int_{-\infty}^{+\infty} a \cdot p(a)da$$

E is a linear operator:

- G(a) = f(a) + h(a) → E(G) = E(f) + E(h)
- G(a) = k · h(a) → E(G) = k · E(h)

Moments of a distribution

The features of a probability distribution can be summarized by few key quantities

$$\mu = E\{a\}, \qquad M_n = E\{|a - E\{a\}|^n\}$$

- Variance (n=2) → spread around μ

$$\sigma^2 = E\{|a - E\{a\}|^2\} = E\{a^2\} - E\{|E\{a\}|^2\}$$

- Skewness (n=3)→ asymmetry

$$\gamma = E\{|a - E\{a\}|^3\}$$

- Kurtosis(n=4)→ tails

$$k = E\{|a - E\{a\}|^4\}$$

Autocorrelation and Autocovariance

- If *a(x)* is a complex random variable expressed as a function of *x*, the autocorrelation of *a(x) is defined as follows:*

$$R\{x', x' + x\} = E\{a(x')\bar{a}(x' + x)\}$$

The autocorrelation describes the correlation of $\boldsymbol{a(x')}$ with itself at a location displaced by *x*

- Similarly, the autocovariance is defined as follows

$$K\{x', x' + x\} = E\{\Delta a(x')\overline{\Delta a}(x' + x)\} = R(x', x' + x) - E\{a(x')\}E\{\bar{a}(x' + x)\}$$

Random events for X-ray photon counting detectors (PCD)

Photon counting detectors

A PCD counts the number of incident photons

0	+1	0	0
0	+1	0	0
0	0	0	+1

Main properties
linearity
shift-invariant response
stationary response

If the detector matrix is illuminated by photon beam with an uniform brightness such that the average number of photons per element is *N*. The proportion of pixels in a large matrix receiving *k* photons is governed by the **Poisson** distribution:

$$p(k) = \frac{N^k e^{-N}}{k!}$$

With $\mu = N$ and $\sigma = \sqrt{N}$

Wide sense stationary (WSS) random processes

If *a* is a real random variable for which the expected value and variance are stationary, the expected value *E{a}* is given by the sample mean:

$$\mu = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} a_m \; ; \qquad \sigma^2 = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} (a_m - E\{a_m\})^2$$

Example:

$Input\ number\ of\ photons\ N = 30$

For a given pixel, different acquisitions are different realization of the same random process

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
Pixel record →	30	34	28	39	37	32	31	24	24	36	30	26

Expected values
$\mu = 30.9;\ \sigma = 4.9$

- WSS processes (Stationary expectation value and autocorrelation):

$$R(x', x' + x) = R(x), \qquad K(x', x' + x) = K(x)$$

Ergodic WSS process

Ergodicity: 'expected values can be determined equivalently from *ensemble* averages or *spatial* averages'

Example

$Input\ number\ of\ photons\ N = 30$

Spatial average

30	36	31	39
26	24	32	28
30	24	37	34

Expected values
$\mu = 30.9;\ \sigma = 4.9$

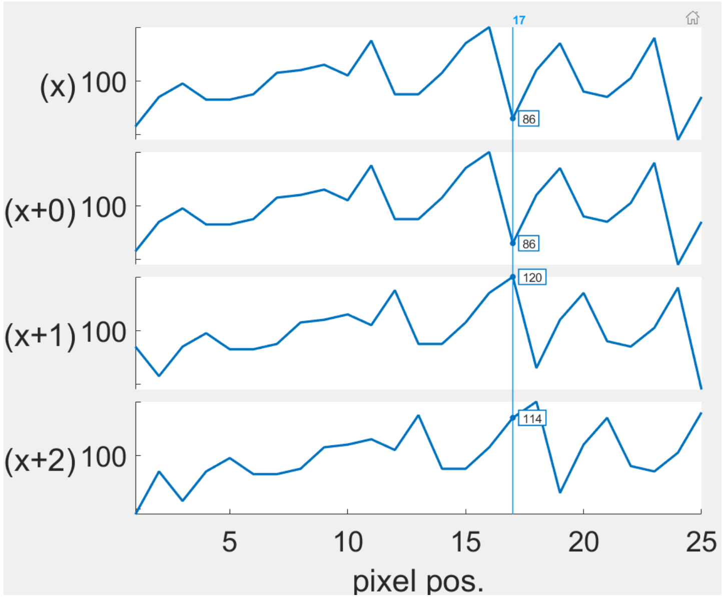
Autocorrelation and Autocovariance interpretation

Ideal case

Each pixel works independently

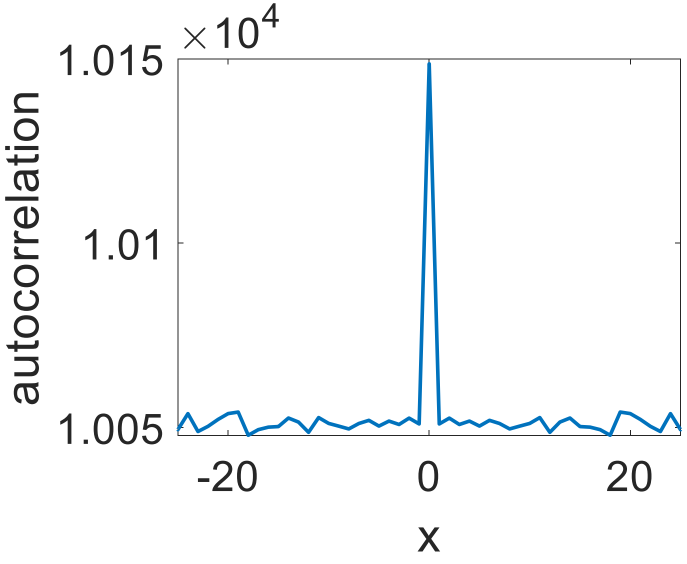
0	0	0	+1	0	0	0

Random Poisson distr. N=100



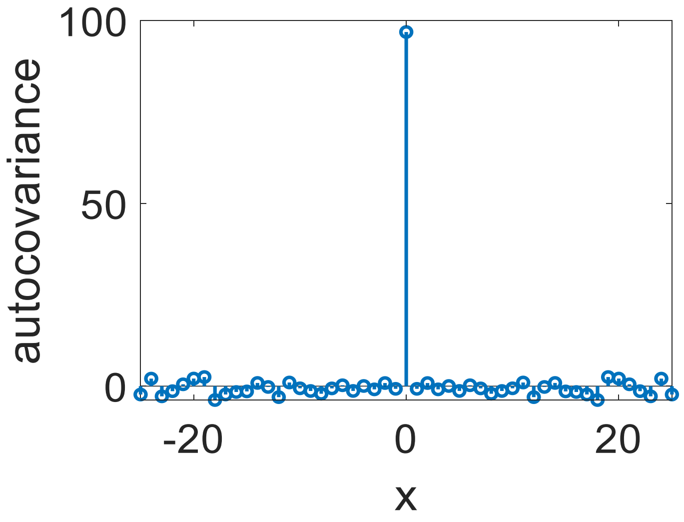
Autocorrelation

$R\{x\} = E\{a(x')\bar{a}(x'+x)\}$



Autocovariance

$K\{x\} = R\{x\} - E\{a(x')\bar{a}(x'+x)\}$

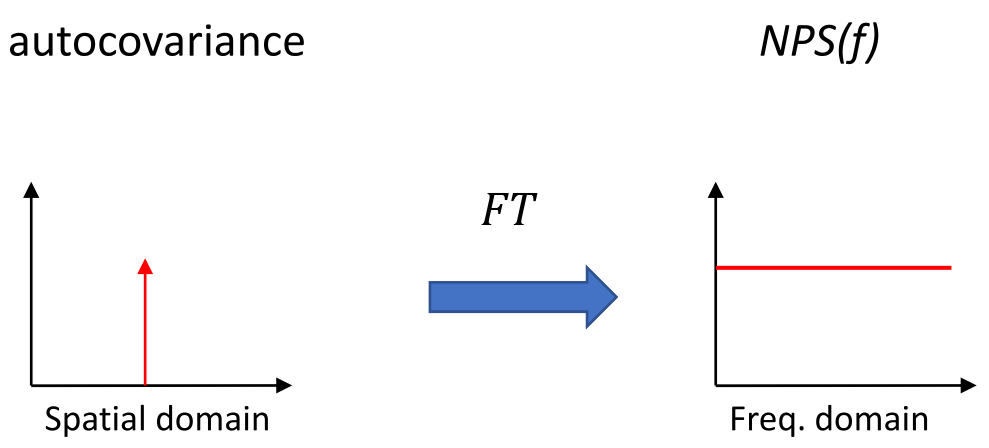


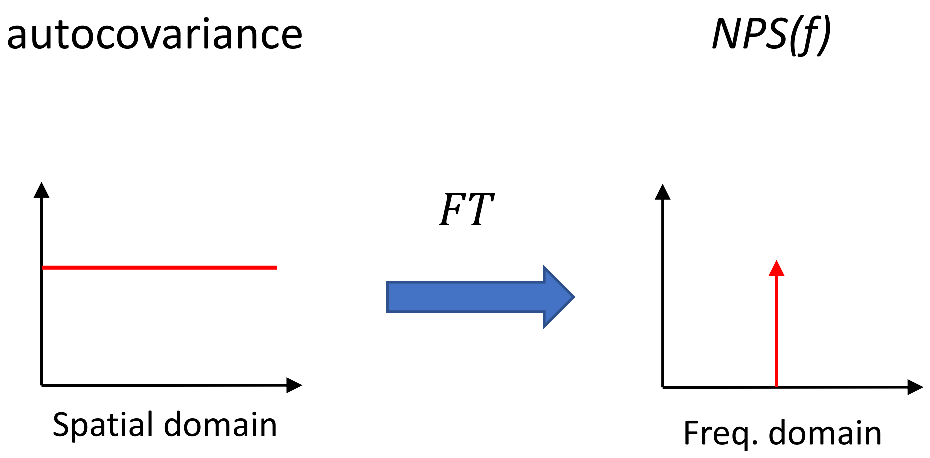
$\sigma^2 = K(0)$

Wiener theorem: noise power spectrum (NPS)

The autocovariance of a WSS random process, ***K(x)***, provides a complete description of the second-moment statistics in the spatial domain. In the spatial-frequency domain, the same statistics are described by the Wiener spectrum, equal to the Fourier transform of the autocovariance function.

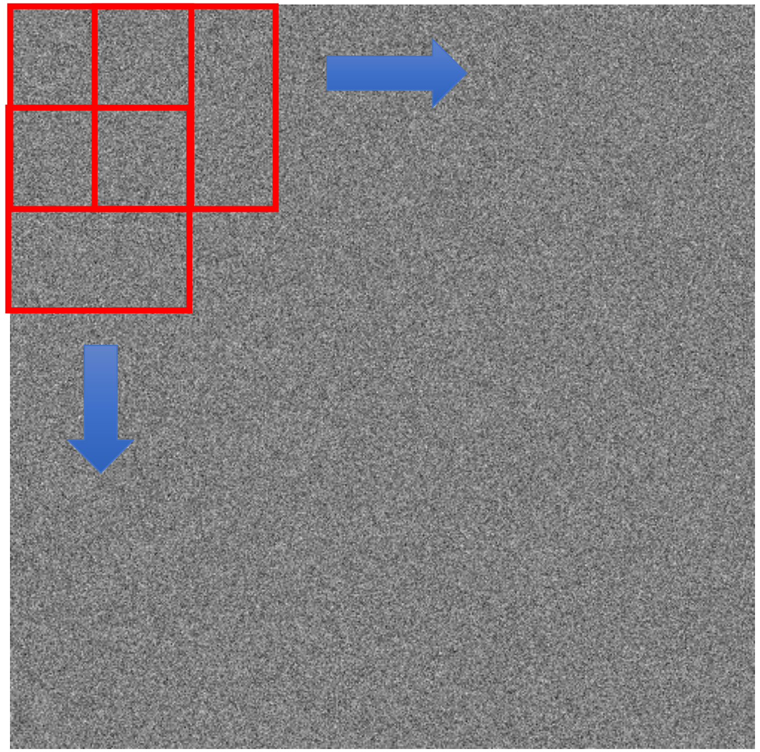
$$NPS(f) = FT\{K(x)\} \rightarrow \sigma^2 = \int NPS(f)df$$

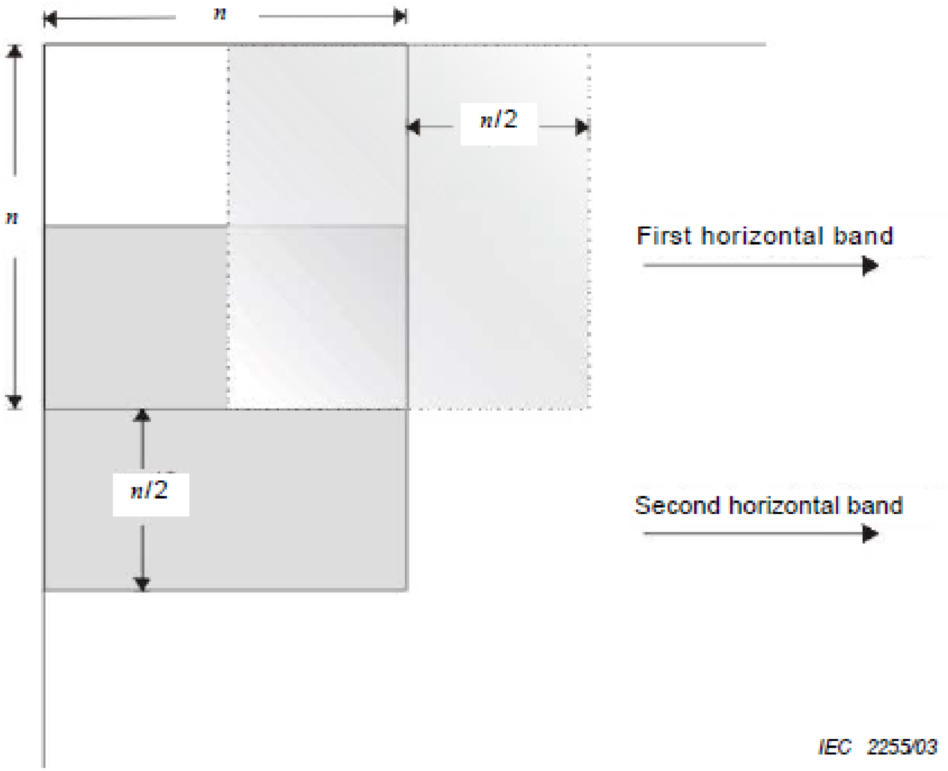




How to measure NPS and autocovariance

A large number of ‘samples’ (→ ∞) is required

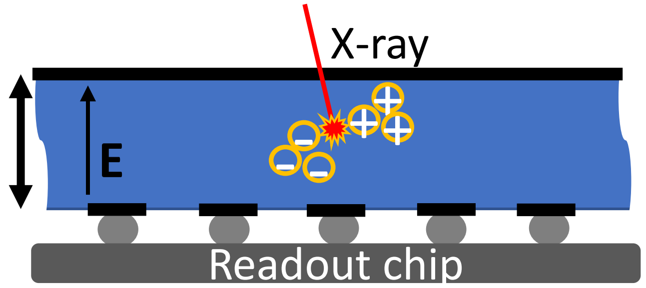


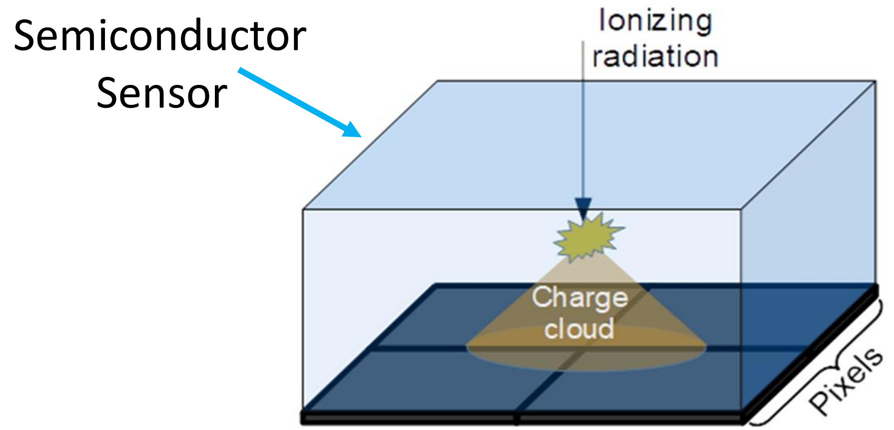


Simulation study

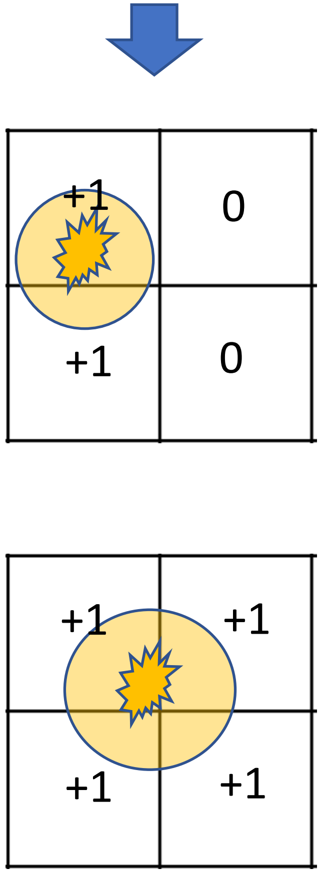
Source of correlations: charge sharing

Direct detection PCD

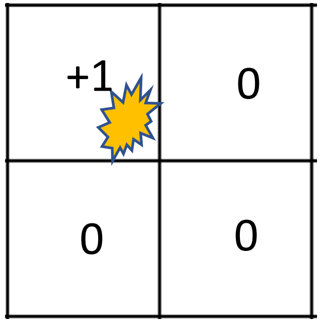




Charge sharing



No Charge sharing



Monte Carlo simulation of multiple counts in PCDs

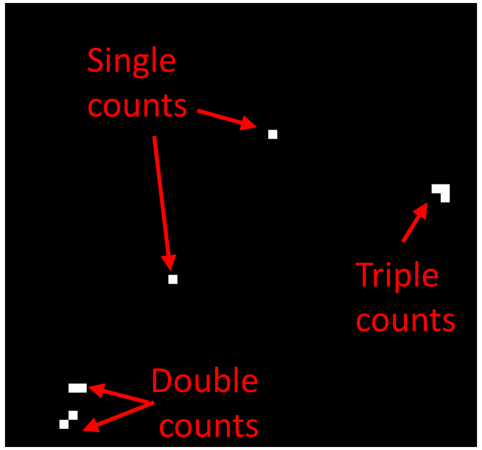
Settings

- PCD (255x255)
- Pixel side $100\mu m$

MC simulation

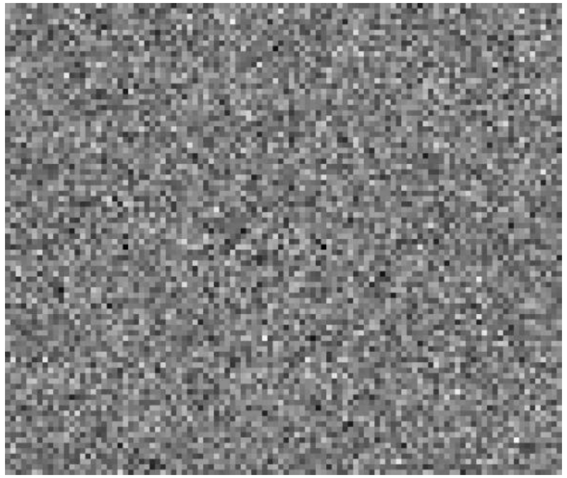
- Each pixel receives N_i photons randomly extracted from a Poisson distribution with $\mu = N_0$
- The percentage of multiple counts (double, triple, etc.) is defined by the user

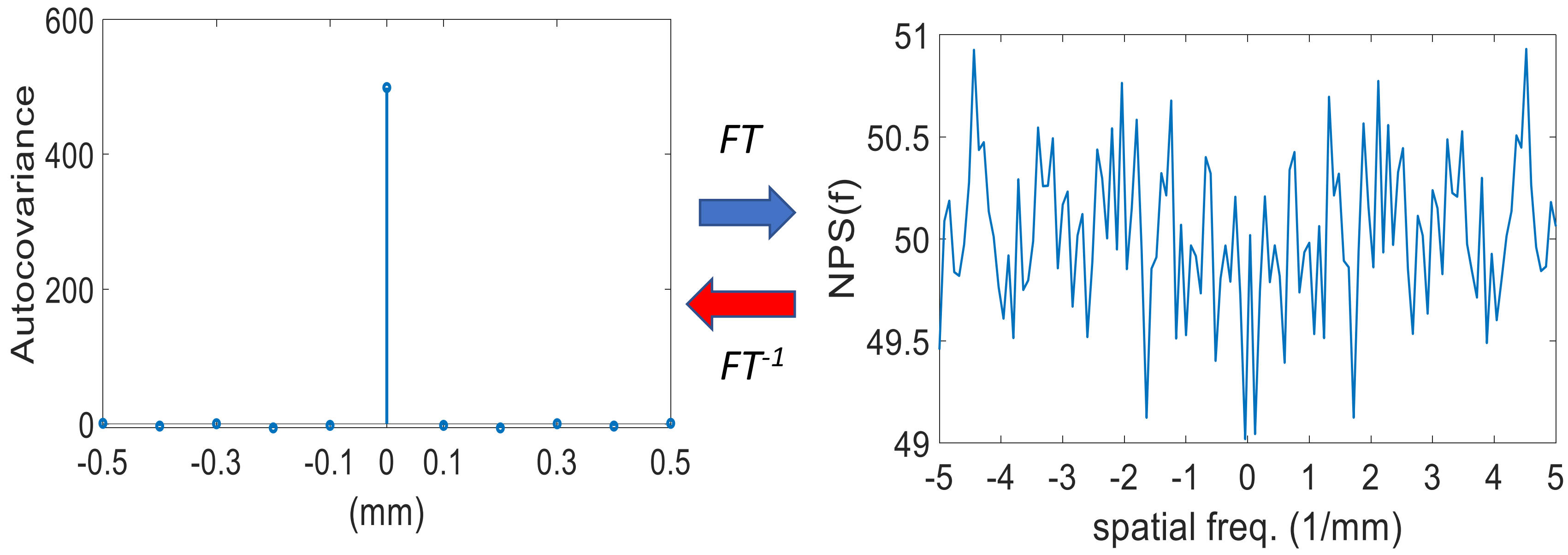
Example of clusters generated by multiple counts (low flow 0.01 photons per pixel)



Case 1

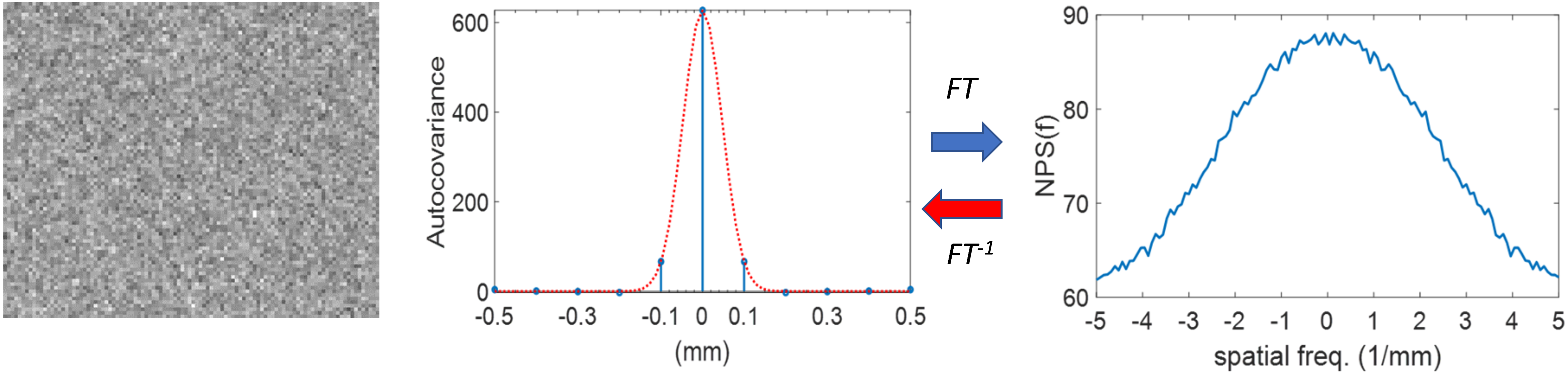
- N_0=500; no multiple counts





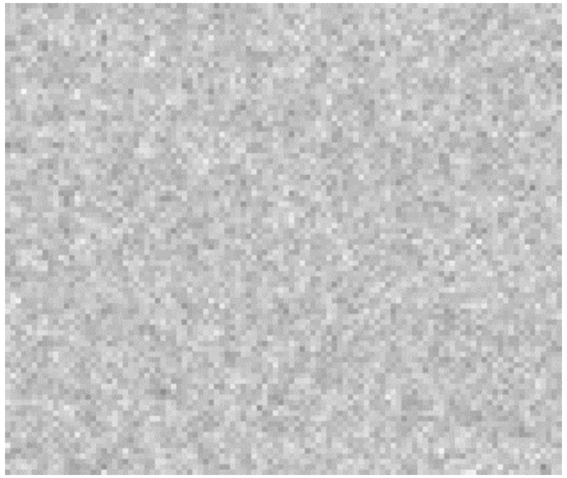
Case 2

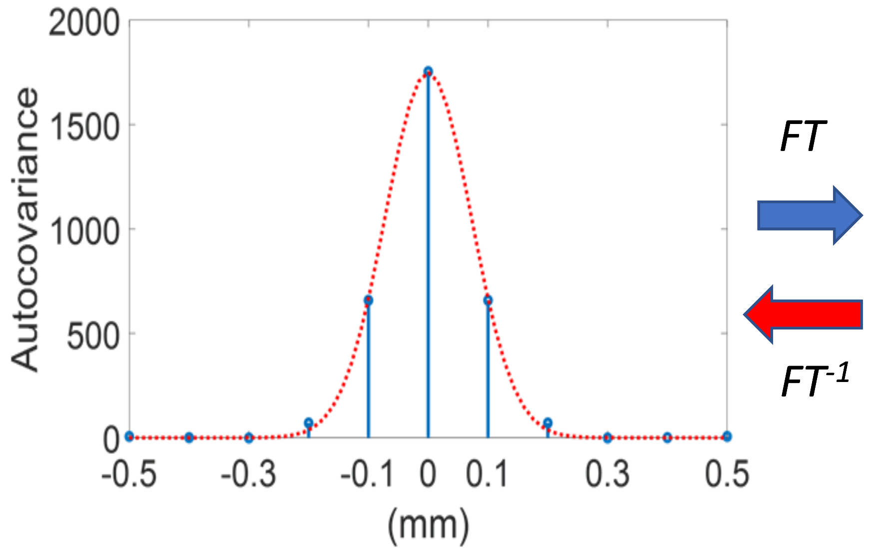
- N_0=500; 50% single counts; 50% double counts

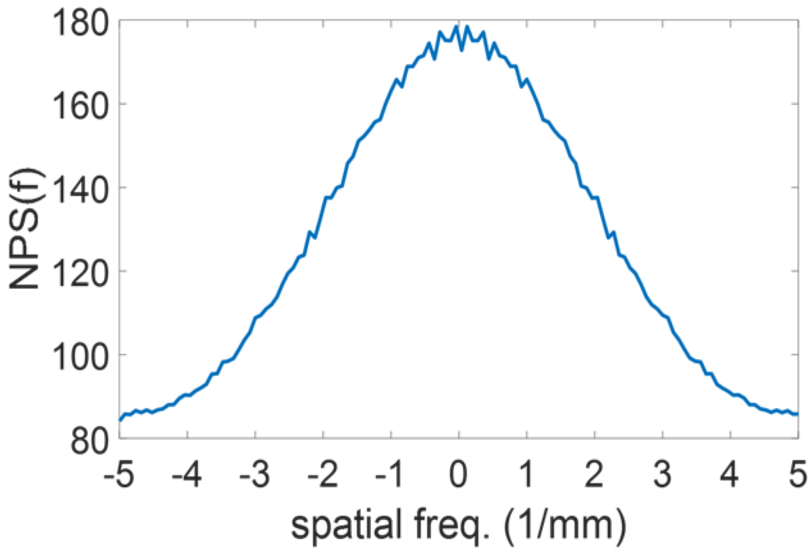


Case 3

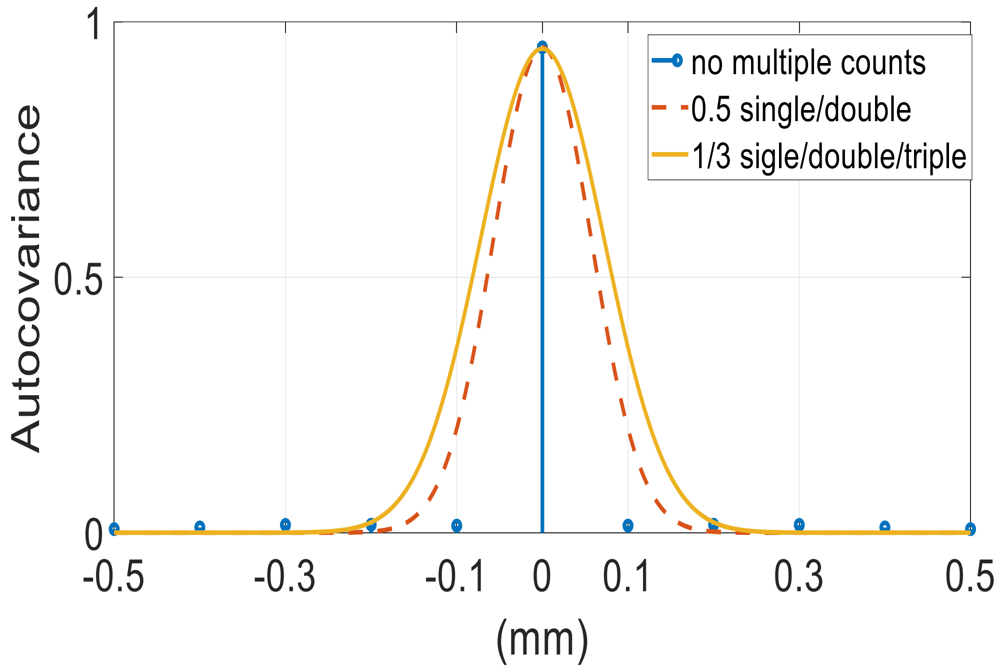
- N_0=500; 1/3 single, double and triple counts



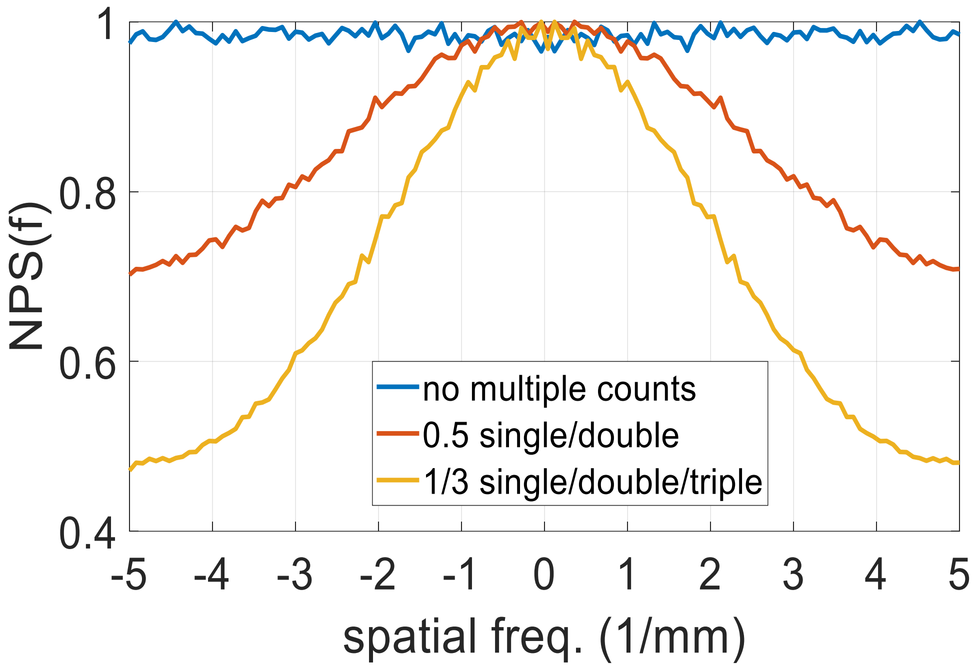




Analysis

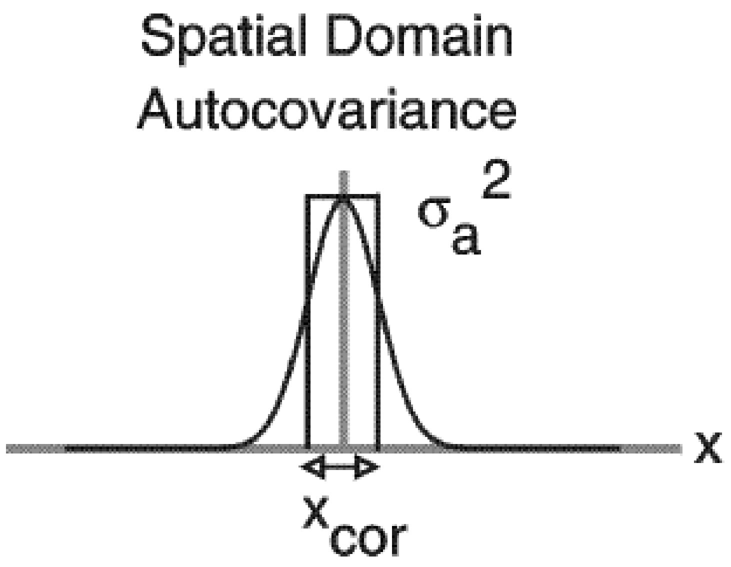


Autocovariance
1
0.5
0
-0.5
-0.3
-0.1
0
0.1
0.3
0.5
(mm)
no multiple counts
0.5 single/double
1/3 sigle/double/triple

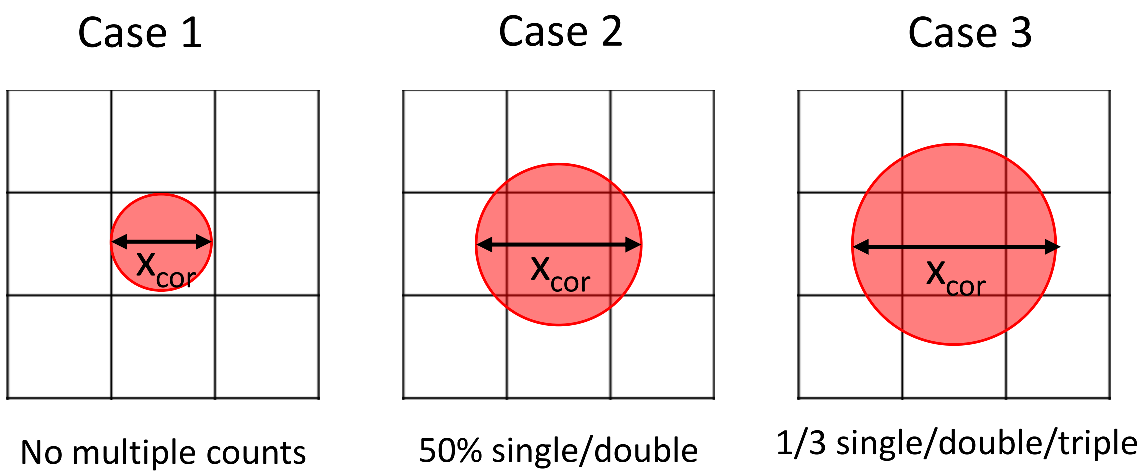


NPS(f)
1
0.8
0.6
0.4
-5
-4
-3
-2
-1
0
1
2
3
4
5
spatial freq. (1/mm)
no multiple counts
0.5 single/double
1/3 single/double/triple

Correlation length



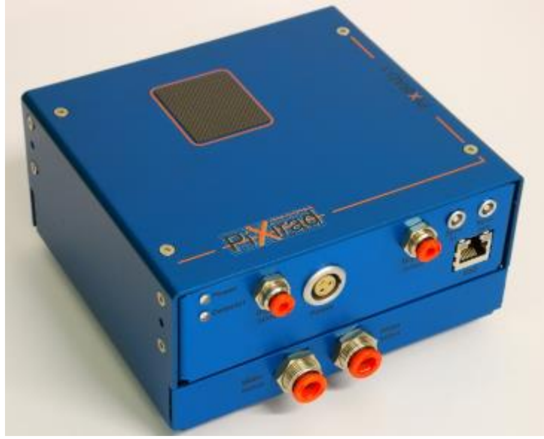
	X_{corr} (μm)
Case 1	100
Case 2	150
Case 3	190



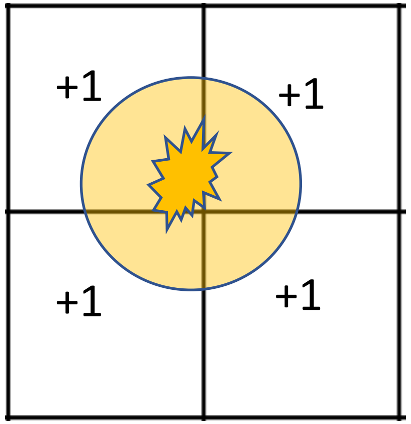
Experimental case

Images acquired with Pixirad-1/pixie-iii

Pixie-iii acquisition modes

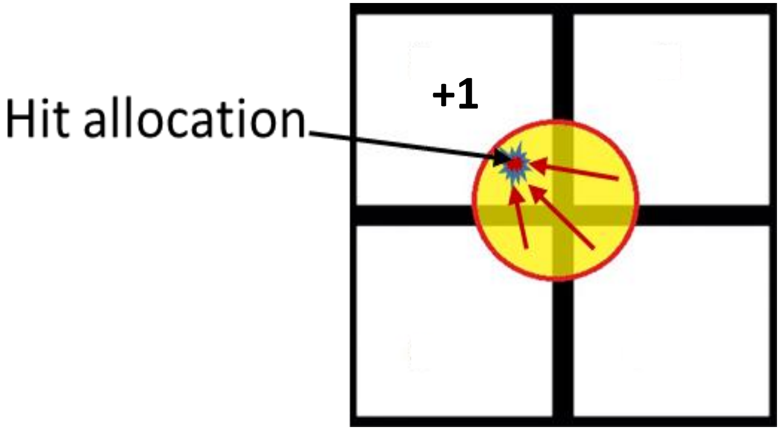


Pixel mode



NPISUM mode

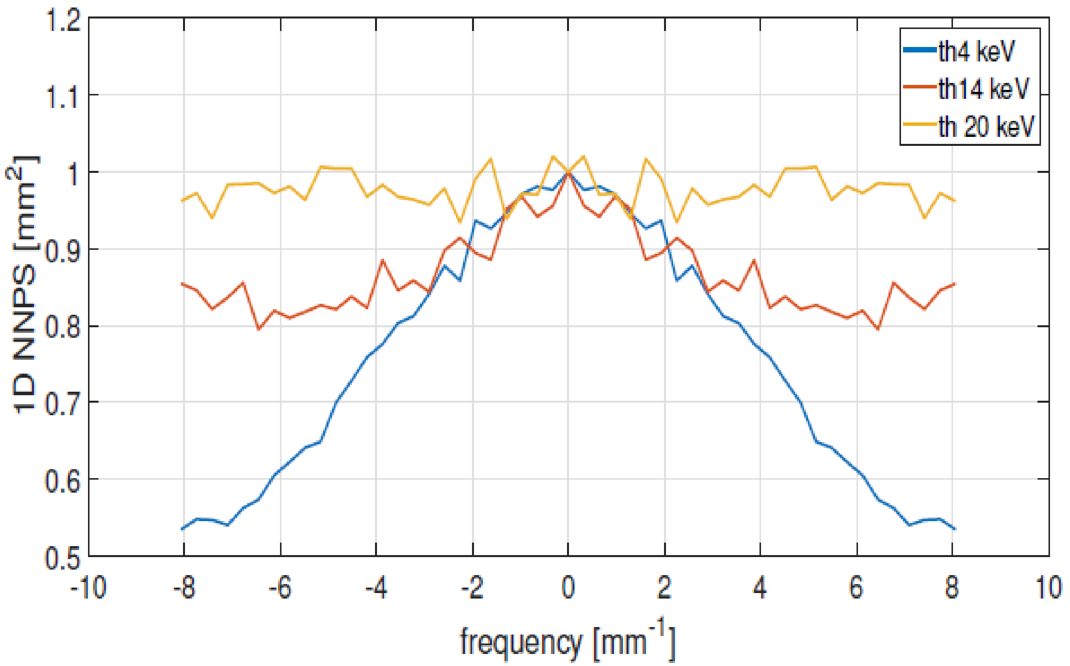
(charge-sharing correction)



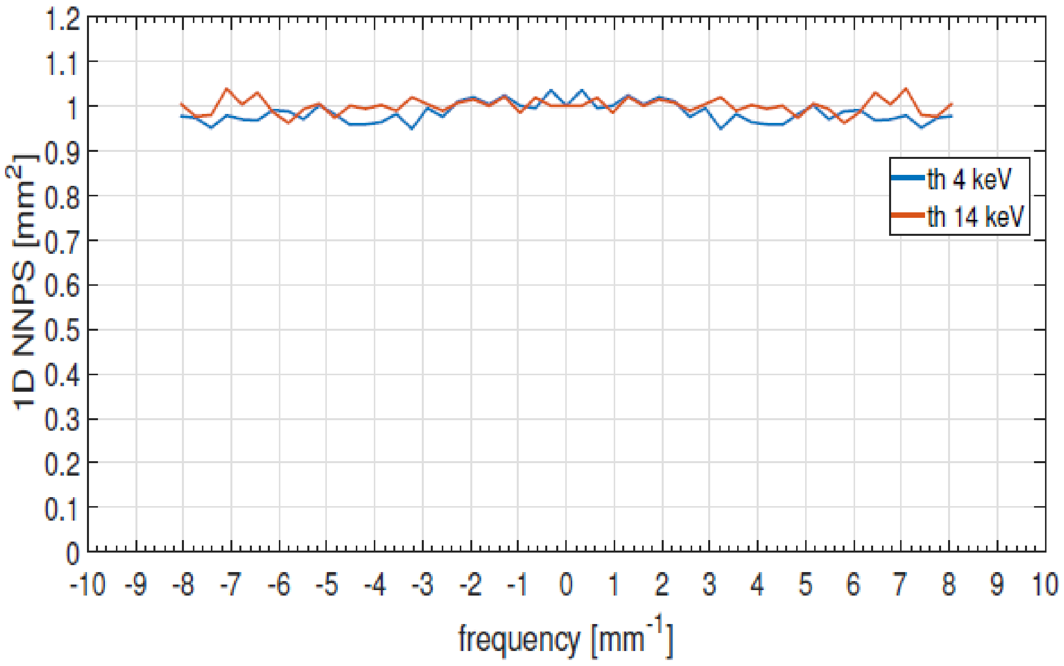
Experimental measurements

Monochromatic beam *E=26 keV*

Pixel mode



NPISUM mode



V. Di Trapani et al., *Characterization of the acquisition modes implemented in Pixirad-1/Pixie-III X-ray Detector: Effects of charge sharing correction on spectral resolution and image quality*, NIM(A), Vol 955 (2020)

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- V. Di Trapani et al., *Characterization of the acquisition modes implemented in Pixirad-1/Pixie-III X-ray Detector: Effects of charge sharing correction on spectral resolution and image quality*, NIM(A), Vol 955 (2020)