Main Methods for Statistical Analysis in gamma-ray Astrophysics

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Statistics Seminar

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della Terra e dell'Ambiente

CONTENTS

WHAT IS STATISTICS?

https://www.wikicasino.it

General definition:

Study of phenomena with unknown results before experimental observations, but showing eventual trend after many of them ("regularity of randomness")

Random variable x

Discrete case

(event probability p_i)

$$\langle x \rangle := \Sigma_i \; p_i \bullet x_i$$

$$\sigma^2 := \langle (x_i - \langle x \rangle)^2 \rangle \quad \ldots$$

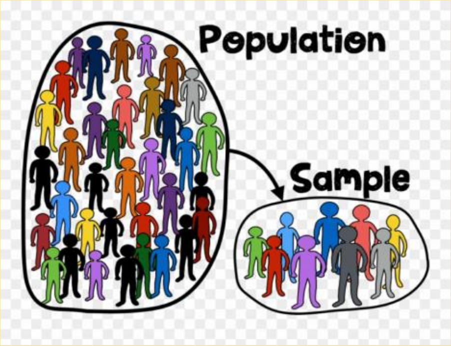
Theory of Probability (assiomatic)

Continuous case

(probability distribution f(x))

$$\langle x \rangle := \int_{\Delta I} x \; f(x) \; dx$$

$$\sigma^2 := \int_{\Delta I} (x - \langle x \rangle)^2 f(x) \; dx \quad \ldots$$



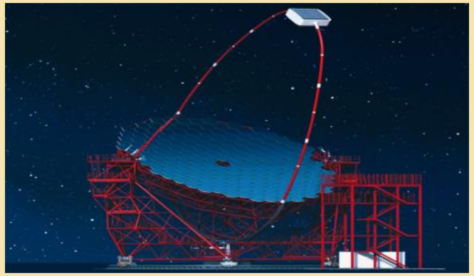
https://www.pngkey.com

APPLICATION TO GAMMA-RAYS

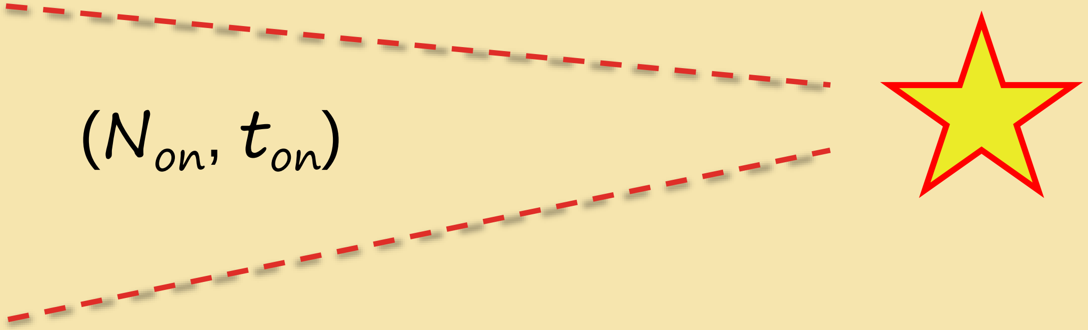
[*LI & MA, APJ, 1983*]

Suppose we are observing on a certain object in the sky, which is thought to be a γ-ray (in particular, photons with energy $E \gtrsim$ GeV) source, for a time t_{on}; then, we can treat the number N_{on} of eventual γ-rays coming from it as a random variable

Artistic Fermi-LAT, Fermi Collaboration



Artistic LST, cta-observatory.org



(N_{on}, t_{on})

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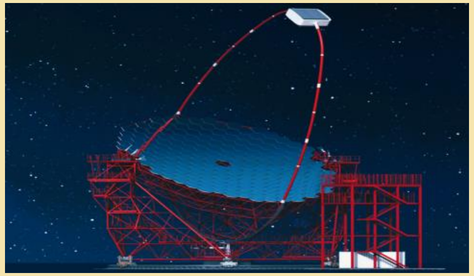

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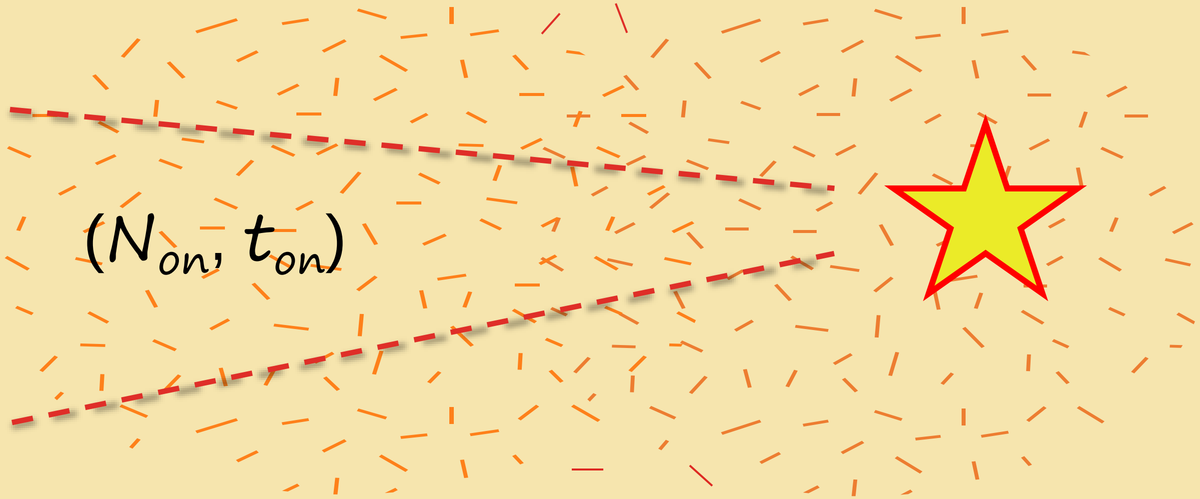

(N_{on}, t_{on})

But in general, also external, spurious signals are being detected!

SOURCE/BACKGROUND ESTIMATION

SNR and instrumental sensitivity often limited, difficult background...

Are our γ sources real?

Statistical reliability := Probability that the observed signal is due only to the background ($\mathcal{B}$) => count rate excess corresponds to a real source $\mathcal{S}$ (or line), not to a different fluctuation

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"On" source pointing

"Off" zone pointing

(N_{on}, t_{on})

(N_{off}, t_{off})

Defining $\alpha := t_{on} / t_{off}$

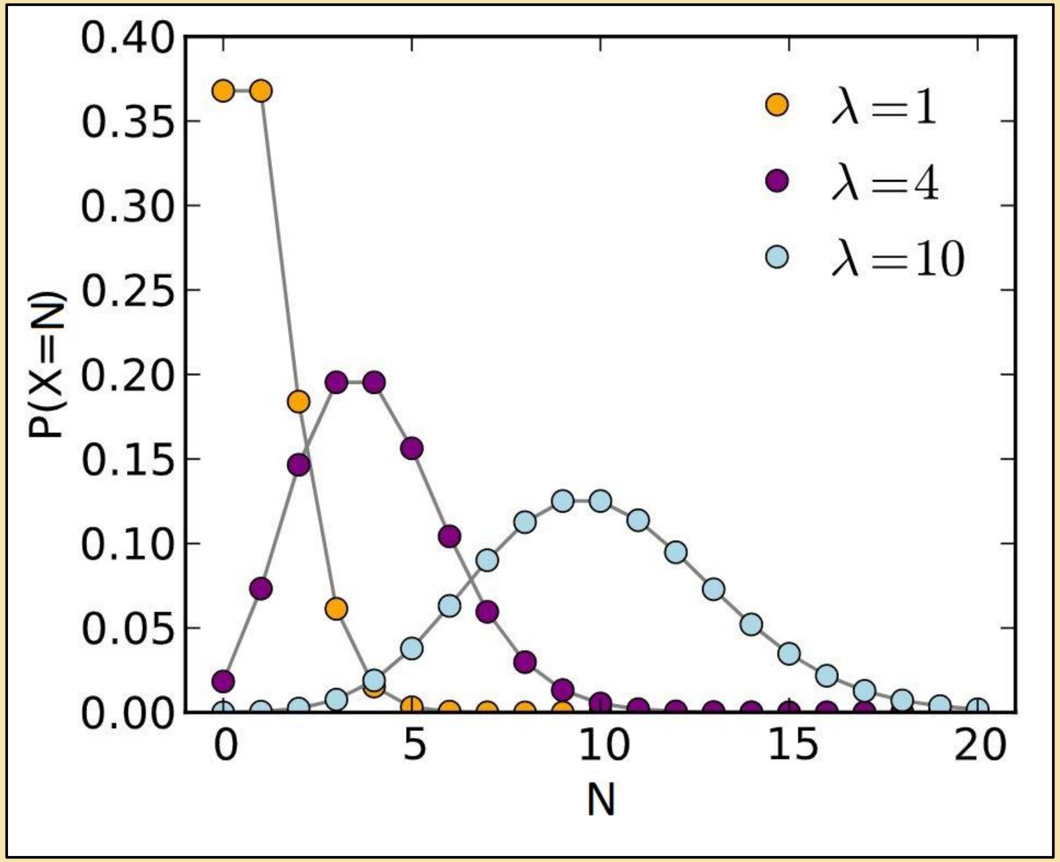
$$N_{\mathcal{B}} = \alpha \bullet N_{off}$$

$$\Downarrow$$

$$\boxed{N_{\mathcal{S}} = N_{on} - \alpha \bullet N_{off}}$$

POISSON DISTRIBUTION

We can describe both N_{on} and N_{off} counts as following a *Poisson distribution*:



From https://www.how2shout.com

$$\boxed{P(\lambda, N) = \frac{\lambda^N}{N!} e^{-\lambda}}$$

$\lambda, N \in \mathbb{N}$

"Small numbers", "Rare events" Law

Discrete probability distribution of N successive, independent events occurring in a certain time or space interval (λ are observed on average)

$<N> = \lambda$ $\sigma^2(N) = \lambda$

STATISTICAL SIGNIFICANCE

$$\mathbb{S} := N_S / \sigma(N_S)$$

<=> "$\mathbb{S}$ standard deviation result"

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$$N_S = N_{on} - \underbrace{\alpha \cdot N_{off}}_{N_B}$$

$$\alpha := t_{on}/t_{off}$$

Poissonian

N_{on}, N_{off}

But MCs...?

$$\mathbb{S}_1 = \frac{N_{\rm on} - \alpha \cdot N_{\rm off}}{\sqrt{N_{\rm on} + \alpha^2 \cdot N_{\rm off}}}$$

$$\mathbb{S}_2 = \frac{N_{\rm on} - \alpha \cdot N_{\rm off}}{\sqrt{\alpha \cdot (N_{\rm on} + N_{\rm off})}}$$

Poissonian

N_{on} ($\sigma^2 = N_B$),

N_{off} ($\sigma^2 = N_B/\alpha$)

Dangerous variants: $N_S / \sqrt{N_B}$ $N_S / \sqrt{N_{on}}$ $N_S / \sqrt{N_S}$

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Poissonian
N_{on} ($\sigma^2 = N_B$),
N_{off} ($\sigma^2 = N_B/\alpha$)

Dangerous variants:

$N_S / \sqrt{N_B}$ — N_{off} only

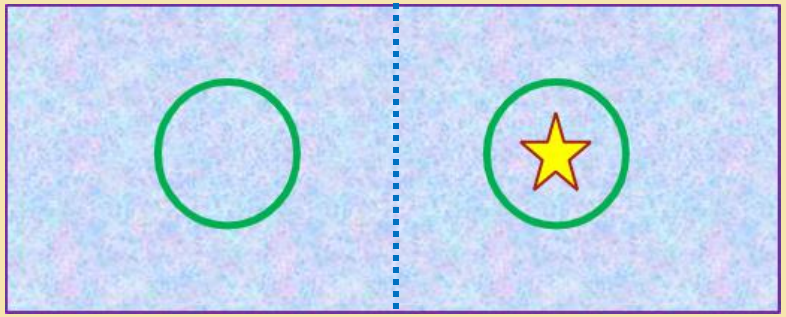
$N_S / \sqrt{N_{on}}$ — N_{on} only

$N_S / \sqrt{N_S}$ — Not (N_{on},N_{off})

→ Under/Overestimation!

SIGNIFICANCE BY HYPOTHESES TEST

$$\mathbb{S} := N_S / \sigma(N_S)$$



Alternative estimation from mathematical statistics:

- **$<N_S>$** and **$<N_B>$** unknown parameters
- **Null Hypothesis** H_0 := "No extra-source, and background only ($<N_S> = 0$)"
  Else, count excess due to a certain γ source =: H_1

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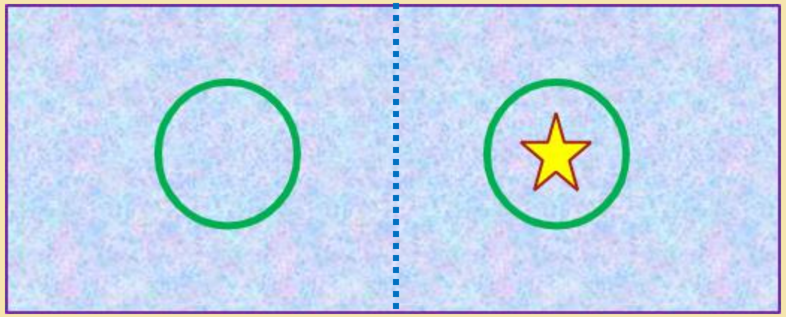

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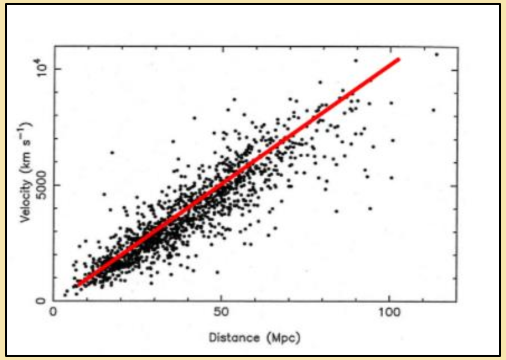
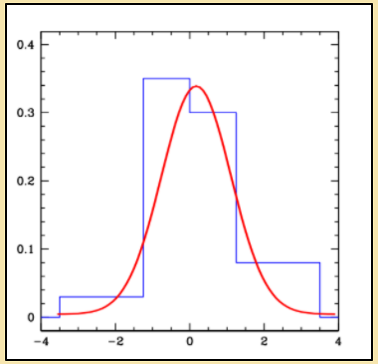
We want to test this to check the existence of some source!

LIKELIHOOD FUNCTION

Given a data set and chosen a model, what's the best estimation of the parameters for a good description?



From "Esperimentazioni di Fisica 1", prof. A. Mucciarelli, Unibo (A.Y. 2020/2021)



"Posterior probability": $P(x_i / A_j) \xrightarrow{\text{independent sampling}} \mathcal{L}(X/\psi) := \prod_{i\ =1,N} P(x_i / A_j)$ $\qquad x_i \in X,\ \ A_j \in \psi$

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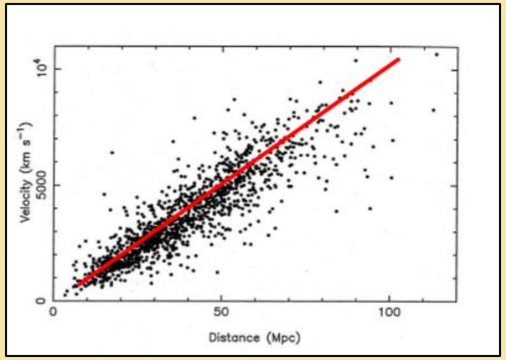

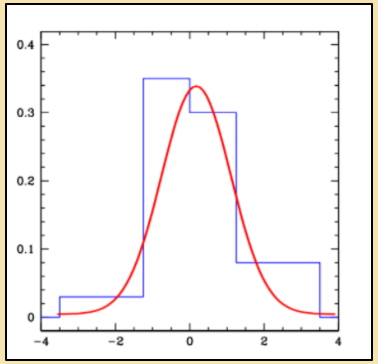

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It is convenient to consider its natural logarithm
The best parameters will be those maximizing it
(***Maximum Likelihood Principle***)

$$\ln \mathcal{L} = \sum \ln P$$

$$\frac{\partial \ln \mathcal{L}}{\partial A} = 0$$

If ψ = (E,U) → (Ê,Û) best estimates; $\hat{U}_c$ conditional for $E=E_0$ case

LIKELIHOOD THEOREM [WILKS 1938]

- Observed data X (N observed values)
- Unknown parameters $\Theta = (H, T)$
- Null hypothesis H_o (and alternative one $H \neq H_o$)
- Maximum Likelihood Ratio $\lambda := \dfrac{\mathcal{L}(X/H_0, \hat{T}_c)}{\mathcal{L}(X/\hat{H}, \hat{T})}$

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If H_o true

Chi Square with *r dof*

$$\boxed{-2 \ln \lambda \sim \chi^2(r)}$$

Asymptotical behaviour

$$N \longrightarrow +\infty$$

Q.E.D.

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Q.E.D.

Let's see how we can use it...

For us:

$X = (N_{on}, N_{off})$, estimated $\Theta = (\langle N_S \rangle, \langle N_B \rangle)$, H_0: $\langle N_S \rangle = 0$ (else H: $\langle N_S \rangle \neq 0$), $T = \langle N_B \rangle$

OUR CASE STUDY...

$X = (N_{on}, N_{off})$, $\Theta = (H = \langle N_S \rangle, T = \langle N_B \rangle)$ with H_0: $\langle N_S \rangle = 0$, $\lambda = \frac{\mathcal{L}(X/H_0, \hat{T}_c)}{\mathcal{L}(X/\hat{H}, \hat{T})}$ => $-2 \ln \lambda \sim \chi^2(r)$

OUR CASE STUDY...

- *HINTS* -

Poissonian counts, H_0 case, $N_{off} = N_B/\alpha$

$X = (N_{on}, N_{off})$, $\Theta = (H = <N_S>, T = <N_B>)$ with H_0: $<N_S> = 0$, $\lambda = \dfrac{\mathcal{L}(X/H_0,\hat{T}_c)}{\mathcal{L}(X/\hat{H},\hat{T})}$ => $-2\ \ln\lambda \sim \chi^2(r)$

* * * * * * * * * * * * * * * * * * * *

$$\mathcal{L}(X/H_0,\hat{T}_c) = \mathcal{P}\left[(N_{\rm on},N_{\rm off})\,/\,<N_{\rm S}>=0,\ <N_{\rm B}>=\frac{\alpha}{1+\alpha}(N_{\rm on}+N_{\rm off})\right] = \mathcal{P}\left[N_{\rm on}\,/\,<N_{\rm on}>=\frac{\alpha}{1+\alpha}(N_{\rm on}+N_{\rm off})\right]\cdot\mathcal{P}\left[N_{\rm off}\,/\,<N_{\rm off}>=\frac{1}{1+\alpha}(N_{\rm on}+N_{\rm off})\right] =$$

$$= \frac{\left[\frac{\alpha}{1+\alpha}(N_{\rm on}+N_{\rm off})\right]^{N_{\rm on}}}{N_{\rm on}!}\ \exp\left[-\frac{\alpha}{1+\alpha}(N_{\rm on}+N_{\rm off})\right]\cdot\frac{\left[\frac{1}{1+\alpha}(N_{\rm on}+N_{\rm off})\right]^{N_{\rm off}}}{N_{\rm off}!}\ \exp\left[-\frac{1}{1+\alpha}(N_{\rm on}+N_{\rm off})\right]$$

$$\mathcal{L}(X/\hat{H},\hat{T}) = \mathcal{P}\left[(N_{\rm on},N_{\rm off})\,/\,<N_{\rm S}>=N_{\rm on}-\alpha N_{\rm off},\ <N_{\rm B}>=\alpha N_{\rm off}\right] = \mathcal{P}\Big(N_{\rm on}\,/\,<N_{\rm on}>=N_{\rm on}\Big)\cdot\mathcal{P}\Big(N_{\rm off}\,/\,<N_{\rm off}>=N_{\rm off}\Big) =$$

$$= \frac{N_{\rm on}^{\rm N_{on}}}{N_{\rm on}!}\ \exp\big(-N_{\rm on}\big)\cdot\frac{N_{\rm off}^{\rm N_{off}}}{N_{\rm off}!}\ \exp\big(-N_{\rm off}\big)$$

GENERAL SIGNIFICANCE

$X = (N_{on}, N_{off})$, $\Theta = (H = \langle N_S \rangle, T = \langle N_B \rangle)$ with H_0: $\langle N_S \rangle = 0$, $\lambda = \dfrac{\mathcal{L}(X/H_0, \hat{T}_c)}{\mathcal{L}(X/\hat{H}, \hat{T})}$ => $-2\ln\lambda \sim \chi^2(r)$

****

$$\lambda = \left[\frac{\alpha}{1+\alpha}\left(\frac{N_{\mathrm{on}}+N_{\mathrm{off}}}{N_{\mathrm{on}}}\right)\right]^{N_{\mathrm{on}}} \cdot \left[\frac{1}{1+\alpha}\left(\frac{N_{\mathrm{on}}+N_{\mathrm{off}}}{N_{\mathrm{off}}}\right)\right]^{N_{\mathrm{off}}}$$

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* *

$$\lambda = \left[\frac{\alpha}{1+\alpha}\left(\frac{N_{\rm on}+N_{\rm off}}{N_{\rm on}}\right)\right]^{N_{\rm on}} \cdot \left[\frac{1}{1+\alpha}\left(\frac{N_{\rm on}+N_{\rm off}}{N_{\rm off}}\right)\right]^{N_{\rm off}}$$

Counts normally (and standardly) distributed if not too few

Dof: *r = 1 (involved $\langle N_S \rangle$ only!)*

✓

$$\chi(1) \sim \sqrt{-2 \ln \lambda} =: \boxed{\mathbb{S}_3 = \sqrt{2}\left\{N_{\rm on} \ln\left(\frac{1+\alpha}{\alpha}\,\frac{N_{\rm on}}{N_{\rm on}+N_{\rm off}}\right) + N_{\rm off} \ln\left[(1+\alpha)\,\frac{N_{\rm off}}{N_{\rm on}+N_{\rm off}}\right]\right\}^{1/2}}$$

CONFIDENCE LEVEL

https://thriveglobal.com

68%

90%

97%

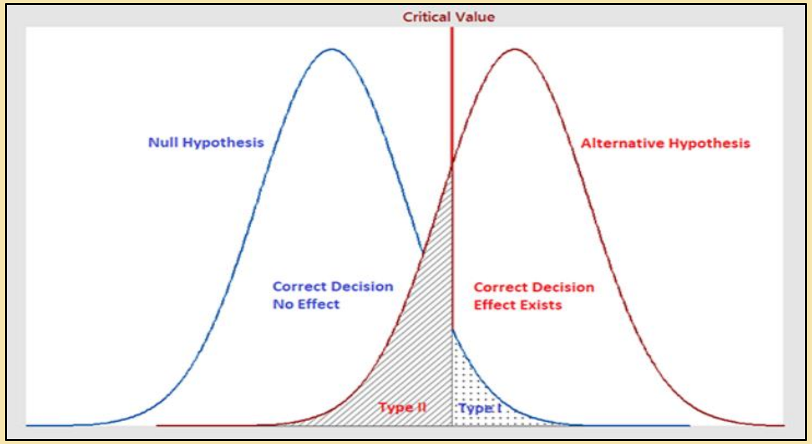
95%

99.99%

99%

H_0 vs H_1

The goal of an observation is to take data, and decide which one of the two has to be chosen according to the result of some Test Statistic (TS) variable → λ



https://statisticsbyjim.com

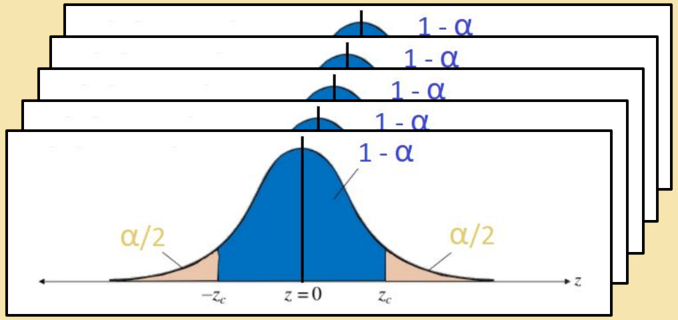
α = type-1 error, false positive (probability of being wrong in rejecting H_0 if true)

To be fixed by experimenters

β = type-2 error, false negative (probability of being wrong in accepting H_0 if false)

Here, if $\langle N_S \rangle > 0$ (or < 0, absorption case) and p is the probability (~ standard gaussian) that an event with at least $\mathbb{S}$ is produced by background → confidence level for a source: $\xi = 1 - p$

MULTIPLE OBSERVATIONS



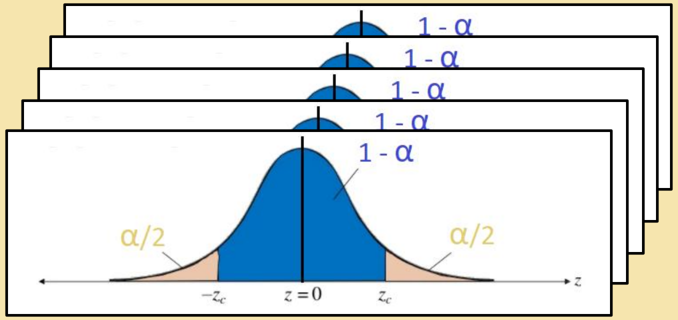
From https://www.slideshare.net

The probability of producing *k* background events with significance ≥ S out of *M* samples will be:

$$P_k = \binom{k}{M} p^k (1-p)^{M-k}$$

*** Binomial Law ***

MULTIPLE OBSERVATIONS



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*** Binomial Law ***

No background case

i.e.

CL for a real source!

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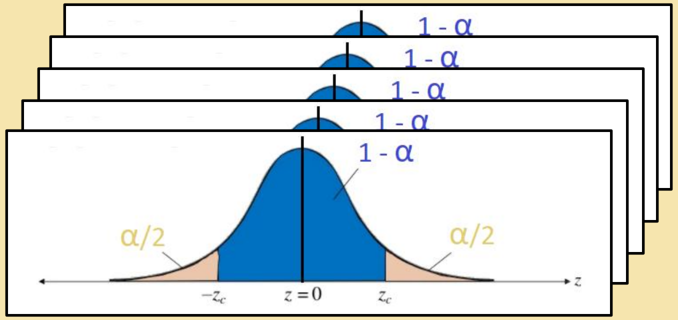

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Alternatives do also exist, e.g. (*Hearn 1969*, *O'Mongain 1973*, *Cherry et al. 1980*):

$$\lambda' := P(N_{on}/B) \,/\, P(N_{on}/S+B) \longrightarrow \xi = 1 - \lambda' \text{ (single)}, \quad \xi = 1 - M\lambda' \text{ (multiple)}$$

But eventual misinterpretation and uncorrectness

MONTE CARLO RESULTS

Random number simulations to check the methods (~10^5 by means of computers)

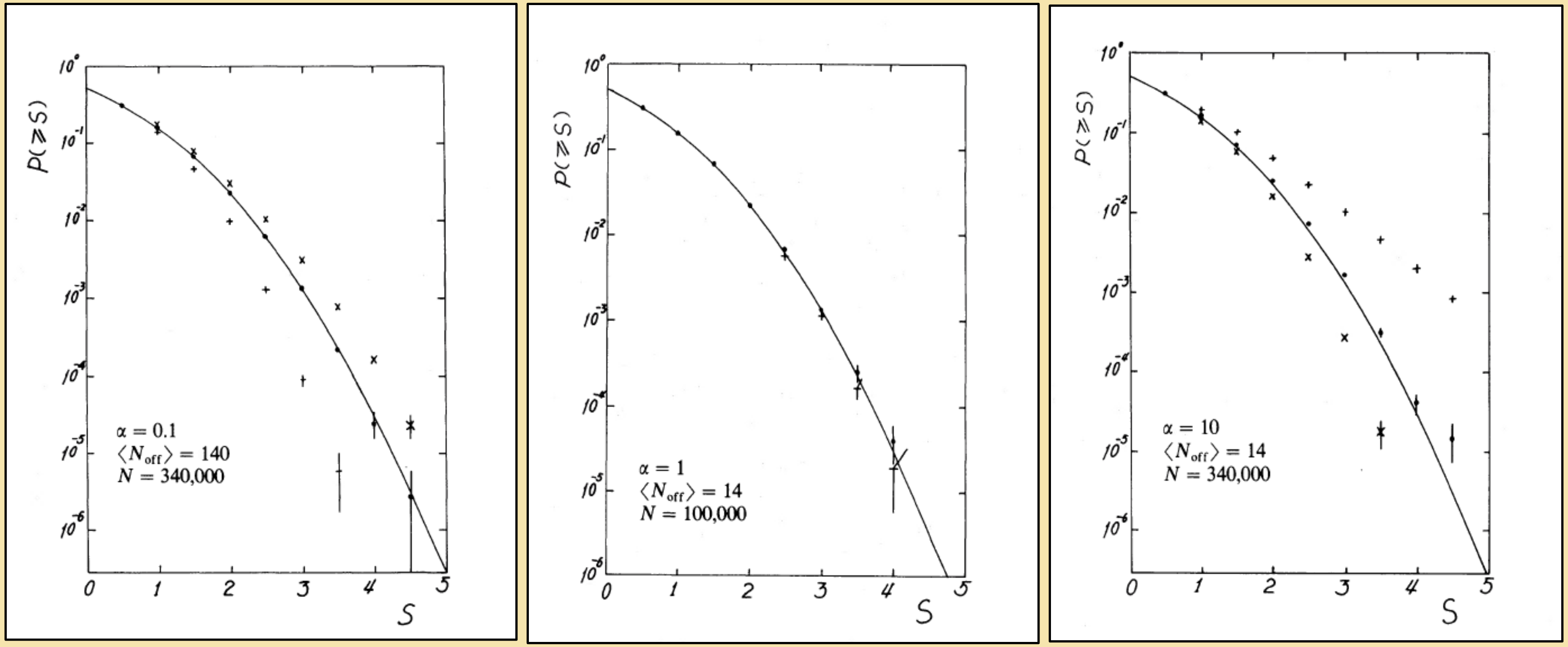
<N_{off}> and α assumed

N_{on} and N_{off} poisson-generated

MONTE CARLO RESULTS

Random number simulations to check the methods (~10^5 by means of computers)

<N_{off}> and α assumed — **N_{on} and N_{off} poisson-generated** » **N_S** » **S**



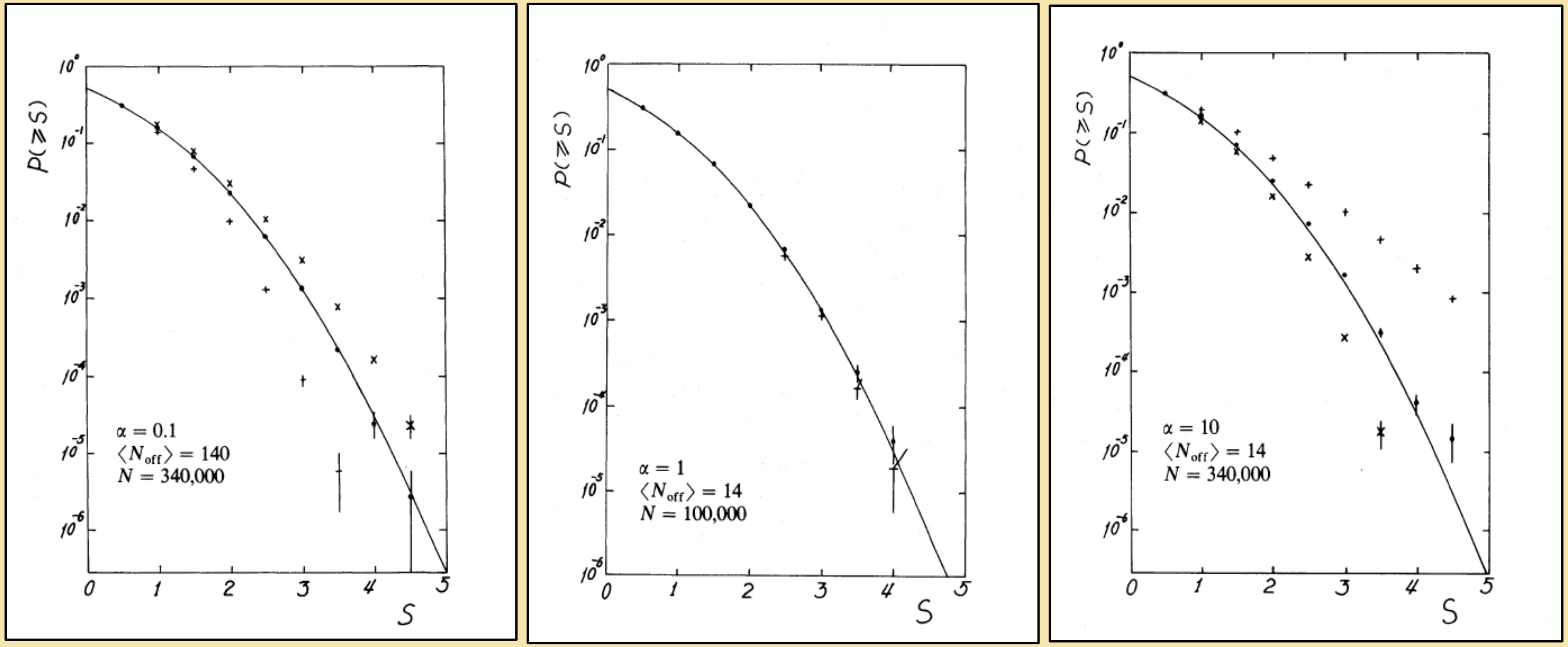
Li & Ma, ApJ, 1983

— Standard Gaussian
✚ $\mathbb{S}_1$
✖ $\mathbb{S}_2$
● $\mathbb{S}_3$

MONTE CARLO RESULTS

Random number simulations to check the methods (~10^5 by means of computers)

$<N_{off}>$ and α assumed — **N_{on} and N_{off} poisson-generated** ≫ **N_s** ≫ **S**



Li & Ma, ApJ, 1983

— Standard Gaussian
✚ S_1
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● S_3

The best

no significant bias!

ANOTHER $\mathbb{S}_3$ CONFIRMATION

Li & Ma, ApJ, 1983

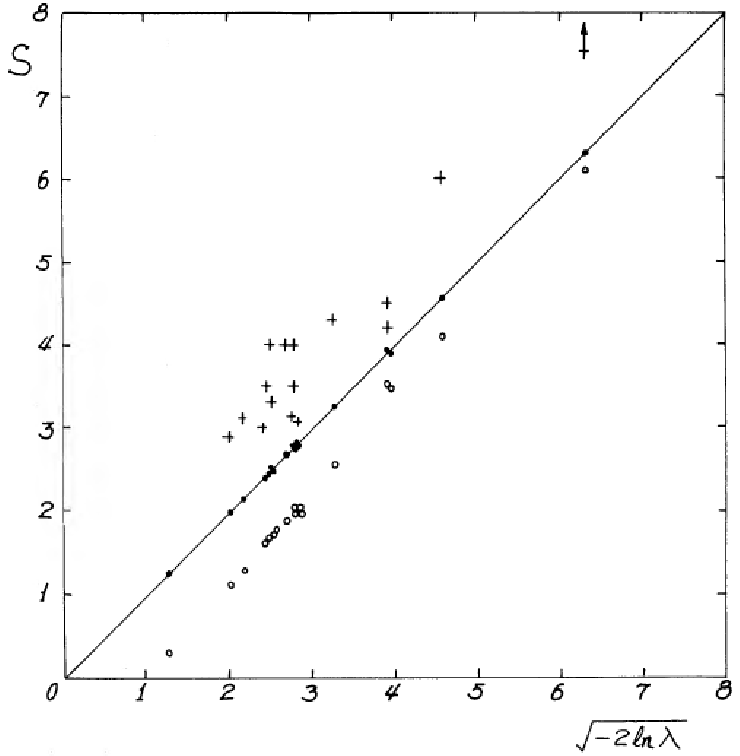

FIG. 3.—Significances of some reported γ-ray lines. *Pluses*, significance S by experimenters; *open circles*, significance S by Cherry *et al.* (1980); *filled circles*, significance S by eq. (17) of this work.

ANOTHER $\mathbb{S}_3$ CONFIRMATION

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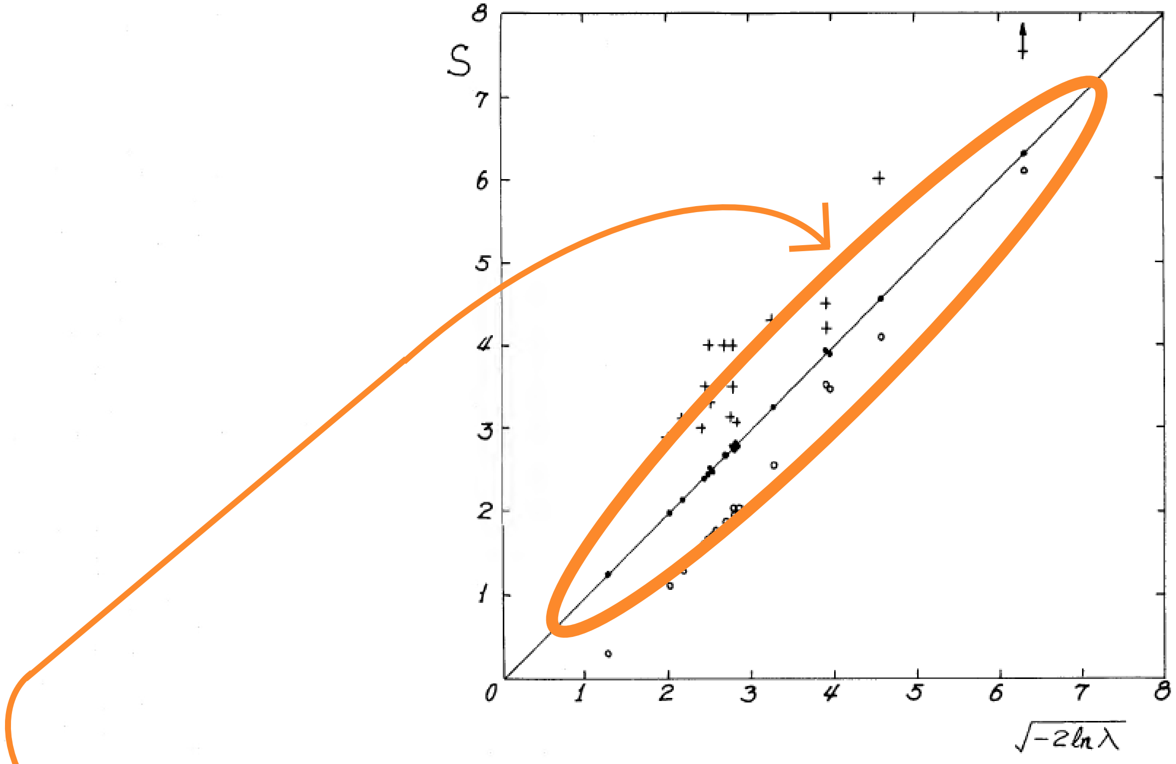

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MAGIC

Major Atmospheric Gamma-ray Imaging Cherenkov

MAGIC Collaboration

- Imaging Atmospheric Cherenkov Telescope system
- La Palma, Canary Islands (29° N, 18° W), at 2,200 m a.s.l.
- Two dishes of 17 m diameter, 85 m aside, with about 250 reflecting mirrors each (M1 since 2004, M2 since 2009)
- Total area ≈ 236 m^2, FoV ≈ 3.5°, angular resolution ≈ 6 arcmin

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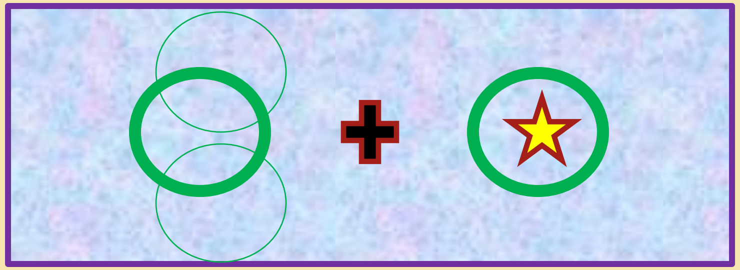
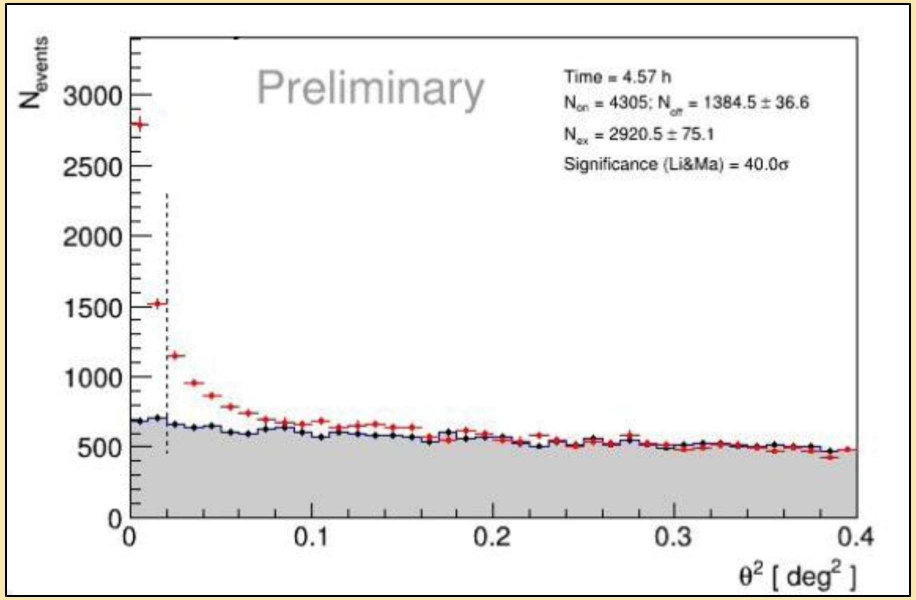
Aiming at indirect detection of primary γ-rays in the ≈ 30 GeV – 100 TeV energy range

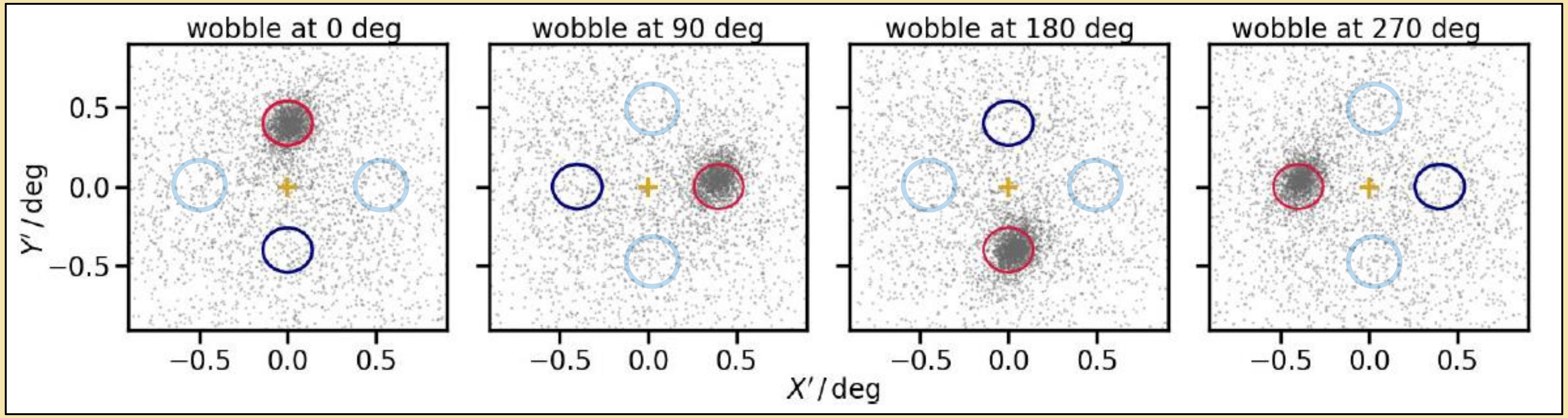


"WOBBLE" mode is the standard (offset ~ 0.4°)!

MAGIC SIGNIFICANCE

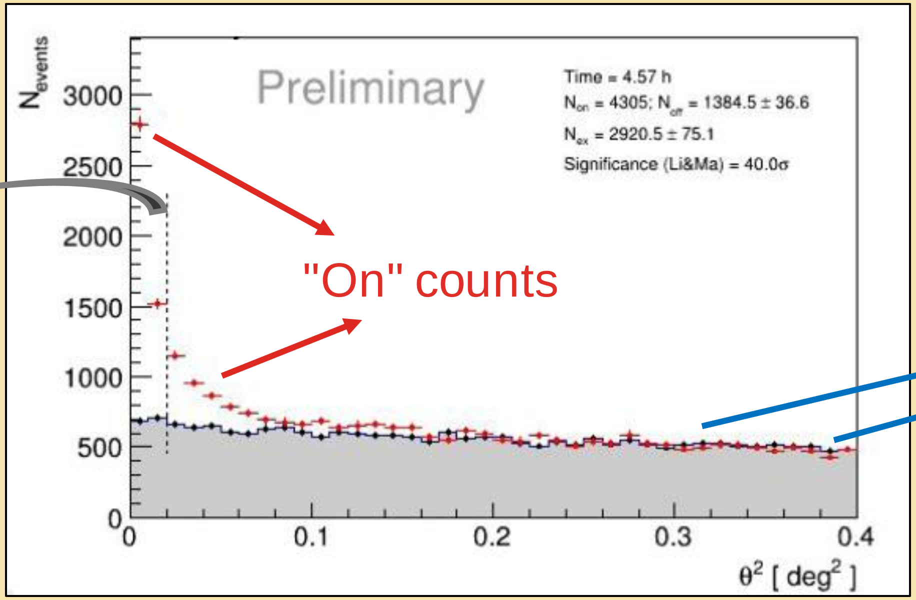


From 8th MAGIC Software School, 2021, Dr. C. Nigro



MAGIC SIGNIFICANCE

MAGIC
Major Atmospheric
Gamma Imaging
Cerenkov Telescopes

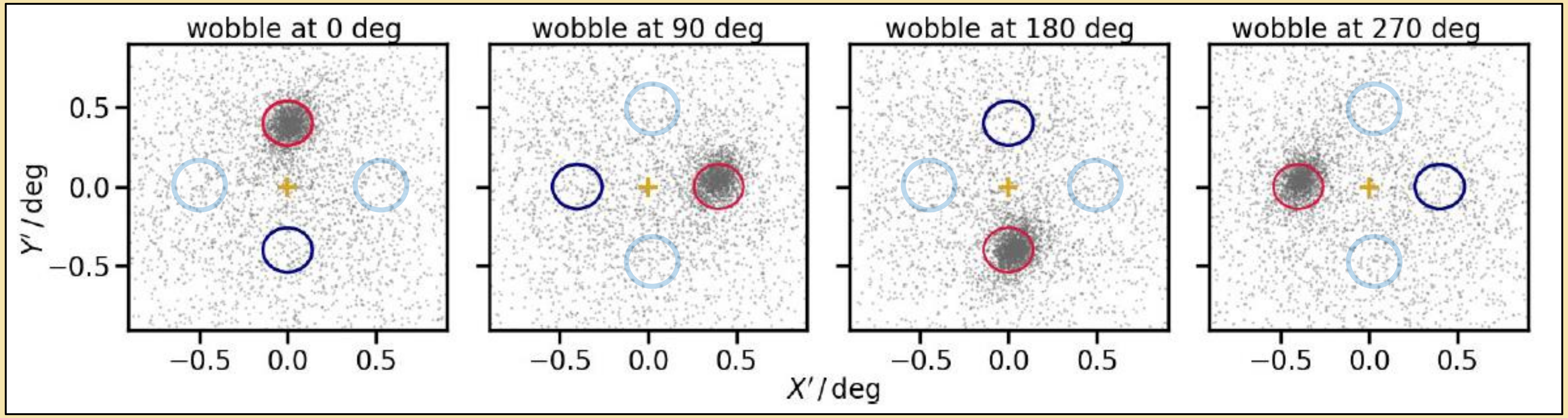


"On" counts

Cut corresponding to
the source limit

"Off" counts
(background level)

From 8th MAGIC Software School, 2021, Dr. C. Nigro



✚ Pointing Position

○ "On" regions

○ "Off" regions
(often more than
one per wobble)

A LI&MA IMPROVEMENT
[VIANELLO, THE ASTROPHYSICAL JOURNAL SUPPLEMENT SERIES, 2018]

What if in our observations some systematic uncertainties (background and/or efficiency) are present?

Simple, yet powerful modelization:

"On" true background (*B) different from "Off" true background (*B*), i.e., $B^* = (k + 1)\,B$**

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Fixed a priori,
or random variable
(it must be > −1;
if = 0, previous case!)

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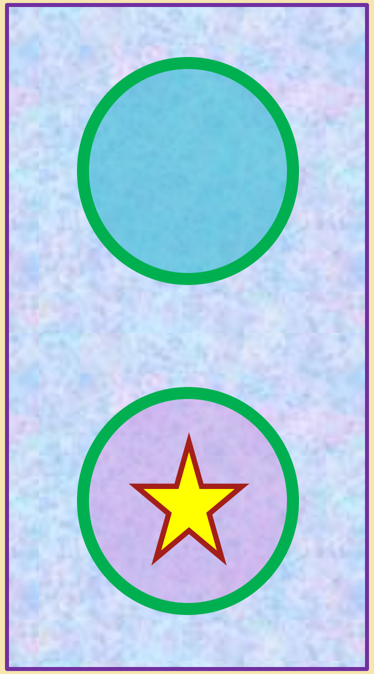


$$P(b|B) = \frac{B^b e^{-B}}{b!}$$

b = "Off" counts

$$P(n|M, B) = \frac{[M + \alpha(k+1)B]^n e^{-[M+\alpha(k+1)B]}}{n!}$$

n = "On" counts
M = Source signal (intensity)

IMPROVEMENT CALCULATION

As before (Wilks Theorem application):

$$P(n, b|M, B) = P(b|B) \times P(n|M, B)$$

Joint probability (source-background independence)

$$\mathrm{TS} = 2\log\left(\frac{\max\{L_1(\theta_1)\}}{\max\{L_0(\theta_0)\}}\right)$$

Test Statistic (*LF* maximized with proper parameters)

$$S = \sqrt{\mathrm{TS}}$$

From χ^2 definition (*dof* difference in H_0 and H_1 is 1)

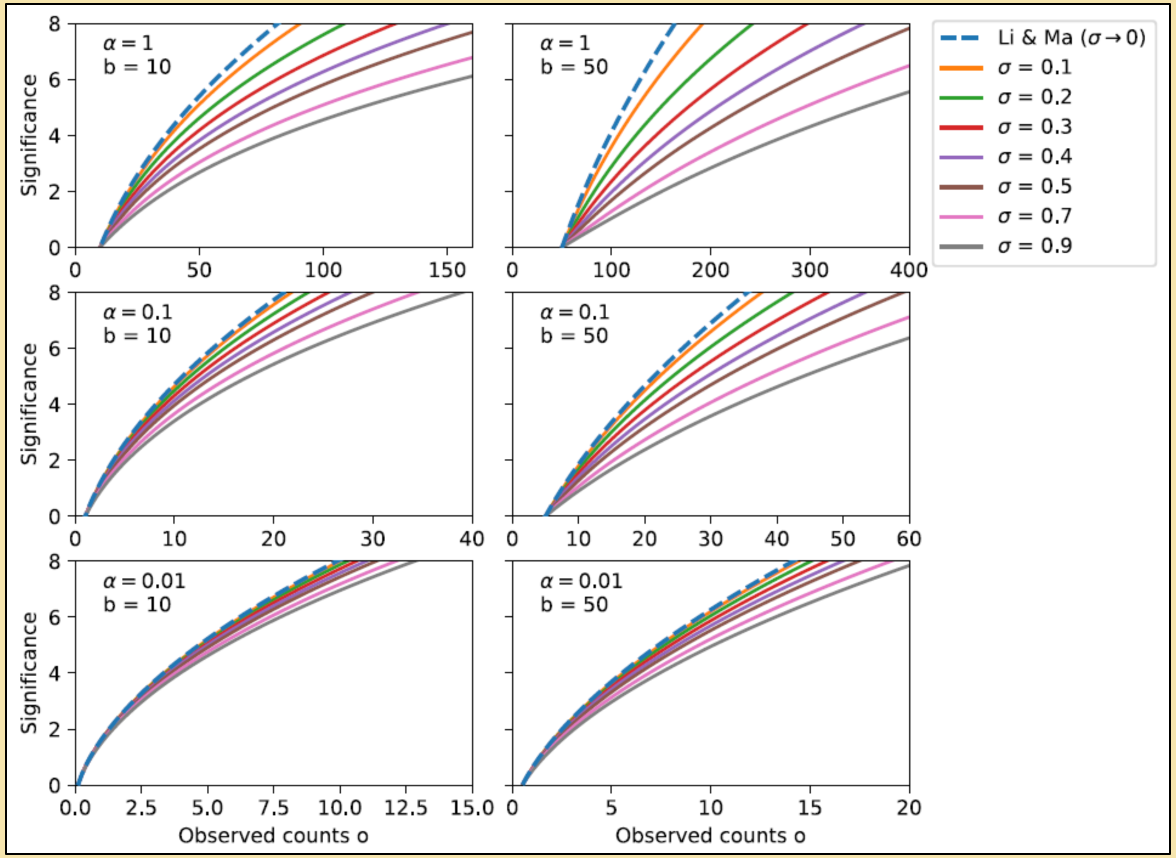
In case k is a random variable, a correspondent factor must be included

If normally distributed,
the missing term is *P(k/σ)*

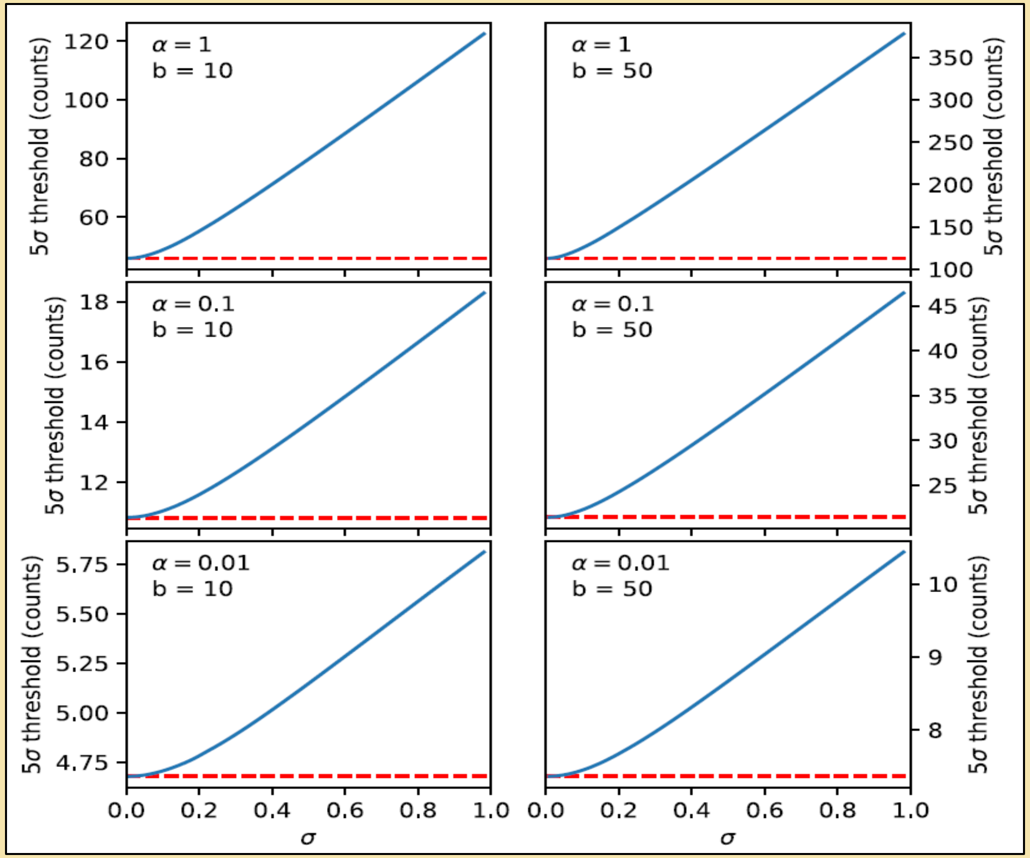
$$2^{1/2}\sqrt{b\log b - b + n\log n - n - \max\{L_0\}}$$

$$\sqrt{2}\left\{n\log\left[\frac{\alpha(k+1)+1}{\alpha(k+1)}\left(\frac{n}{n+b}\right)\right]+b\log\left[(\alpha(k+1)+1)\left(\frac{b}{n+b}\right)\right]\right\}^{1/2}$$

IMPROVEMENT EFFECTS



Vianello, The Astrophysical Journal Supplement Series, 2018



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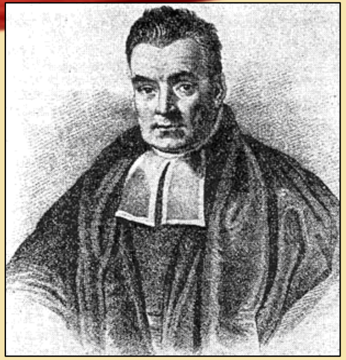
BAYESIAN APPROACH *[GILLESSEN & HARNEY, A&A, 2005]*



Thomas Bayes
(from Wikipedia)

Bayes' Theorem

$$P(H_i/A) = \frac{P(H_i)\, P(A/H_i)}{\sum_{i=1}^{n} P(H_i)\, P(A/H_i)}$$

A = event, $\{H_1, H_2, \ldots, H_n\}$ = hypothesis set

More logical framework with respect to *Li&Ma* classical Statistics:

- **Significance interpreted as probability that a source has intensity > 0**
- **Prior distribution by a simmetry unbiased-result principle, posterior distribution by an intensity parameter λ ("objective procedure")**

⬇

No count limitations, no verification methods (e.g. MCs)!

BAYESIAN RESULTS *[GILLESSEN & HARNEY, A&A, 2005]*

Chosen a probability *K*:

$$\boxed{K(\lambda > \lambda_{\min}) = \int_{\lambda_{\min}}^{\lambda_{\max}} P(\lambda'/n)\, d\lambda'} \xrightarrow{K \approx 1} \boxed{erf\left(\frac{\mathbb{S}}{\sqrt{2}}\right) := \frac{1}{\sqrt{2\pi}} \int_{-\mathbb{S}}^{\mathbb{S}} e^{-x^2/2}\, dx}$$

BAYESIAN RESULTS *[GILLESSEN & HARNEY, A&A, 2005]*

Bayesian interval (error)

Chosen a probability *K*:

$$K(\lambda > \lambda_{\text{min}}) = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda'/n)\, d\lambda'$$

Parameter

Observable

$K \approx 1$

Error Function

Gaussian

$$erf\left(\frac{\mathbb{S}}{\sqrt{2}}\right) := \frac{1}{\sqrt{2\pi}} \int_{-\mathbb{S}}^{\mathbb{S}} e^{-x^2/2} dx$$

Bayesian Significance

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Bayesian Significance

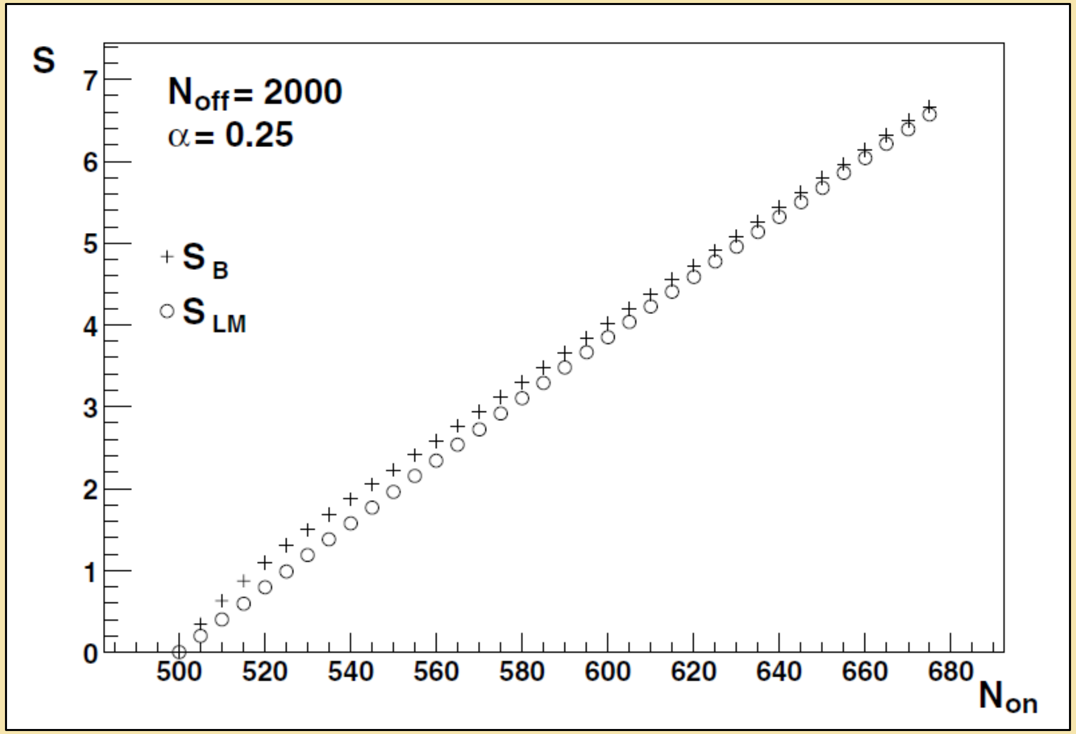
$$\lambda_{on} = \alpha\, \lambda_{off} + \lambda_s$$

Reduction and proper model transformation:

$\Lambda := \lambda_{on} + \lambda_{off}$, $W := \lambda_{on} / \Lambda$, $N := N_{on} + N_{off}$

Total counts → Poissonian , "On"/"Off" → Binomial

Source detection is CL for $\lambda_s > 0$



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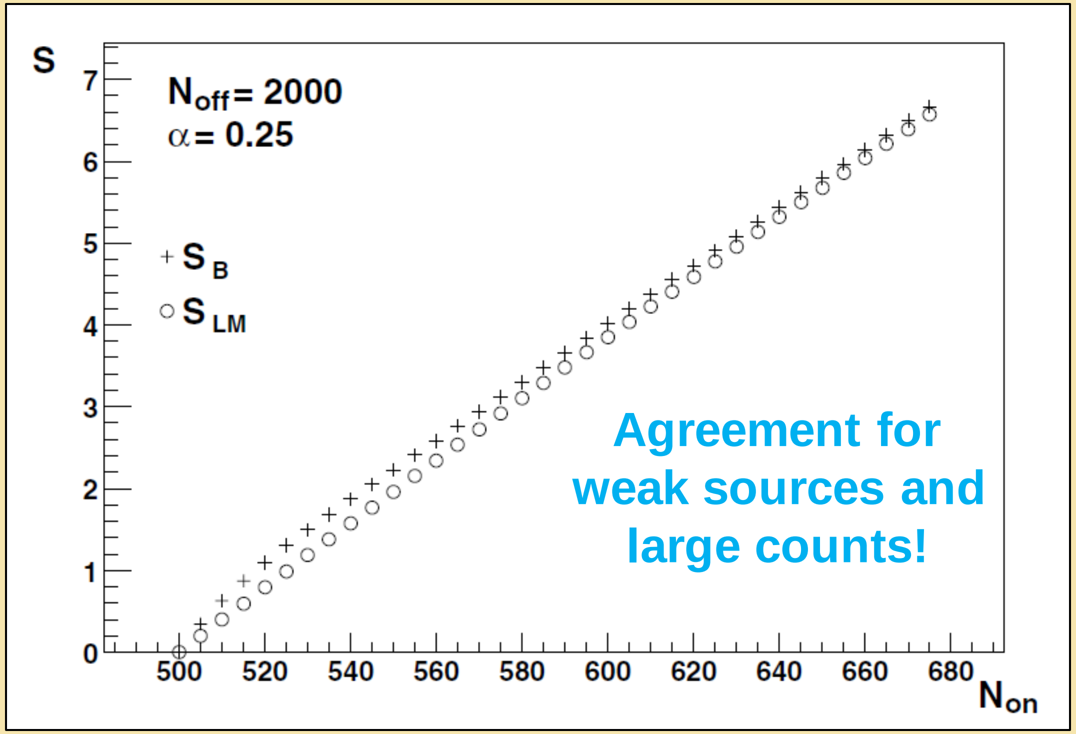

Agreement for weak sources and large counts!

CONCLUSIONS

- ✓ *Gamma-ray observations require statistical reliability methods due to poor source/background separation*

- ✓ ***S_3*** *is the best formula for significance using "On"/"Off" regions and hypotheses test, also for the general $\alpha \neq 1$ case, and not so many counts*

- ✓ *The fundamental Li & Ma significance formula can be improved to also take into account systematic uncertainties often neglected*

- ✓ *Bayesian Statistics could logically be more satisfactory than the frequentist one (agreement for large counts), because of a-priori knowledge (e.g. intensities ≥ 0), no need of verification methods, and correctness for any (N_{on}, N_{off}) photon numbers*

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Thank you for the attention!