The spin variance as a measure of quantum entanglement

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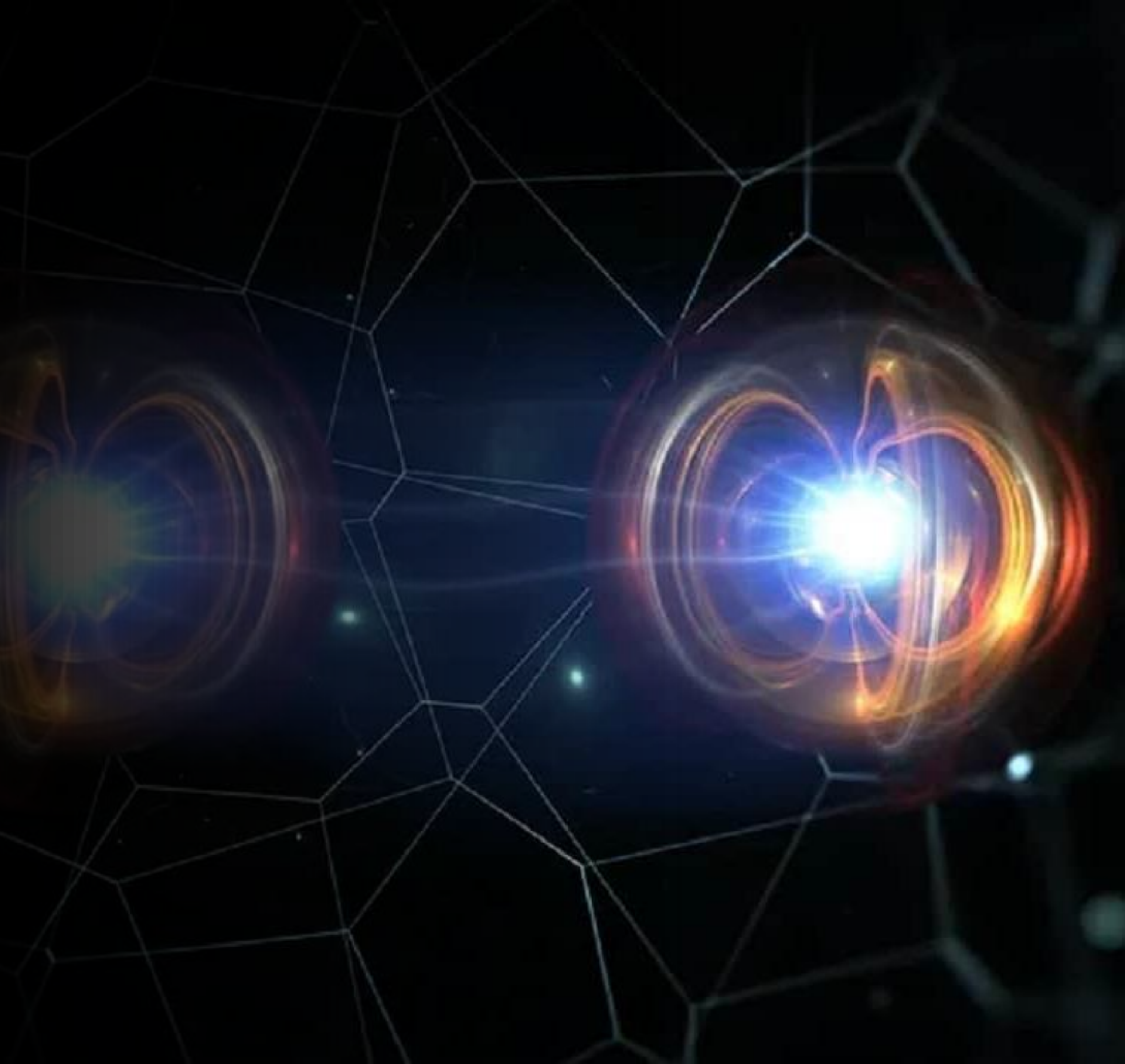
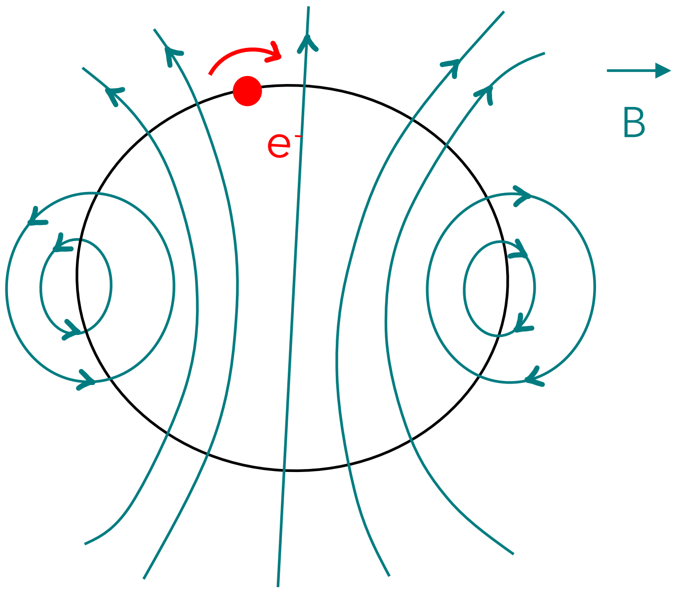
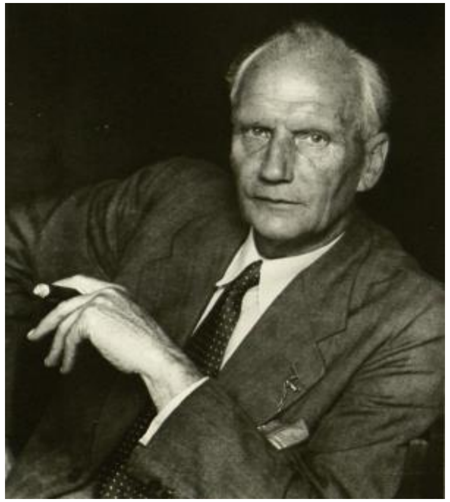


Contents

I- Spin



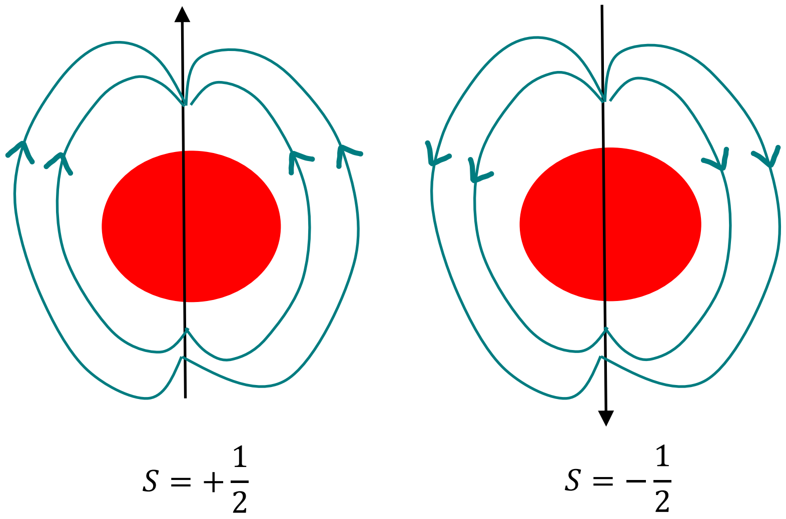
Electromagnetic field due to
the orbital angular momentum



Walther Gerlach

Otto Stern

1920



Electromagnetic field due to
An intrinsic angular momentum

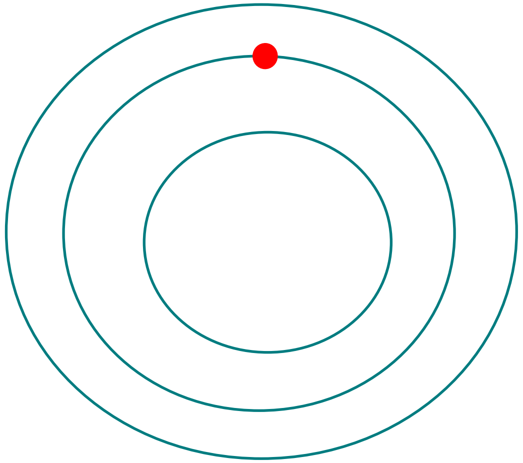
Intrinsic	Extrinsic
Mass	Energy
Charge	Position
Spin	Linear momentum
	Orbital angular momentum

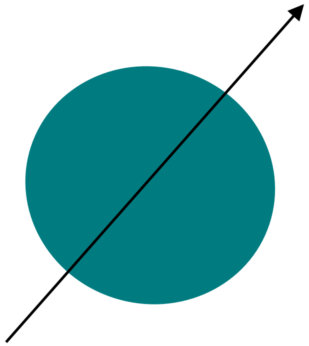
Niels Bohr

1913

Orbital angular momentum l is quantized

$$l = n\hbar$$





$|S, m_s>$

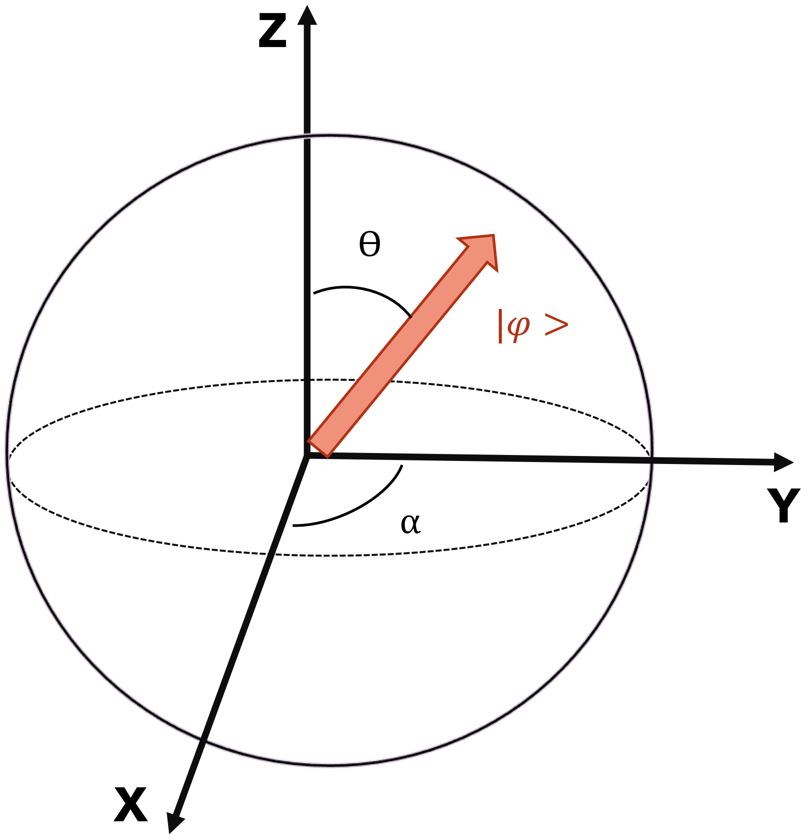
The projection of the spin on any fixed direction is an integer or half-integer multiple of Planck's constant $\hbar$. The only possible values are

$$m_s = -s\hbar,\ (-s+1)\hbar,\ \ldots,\ (s-1)\hbar,\ s\hbar$$

Example: S=1/2

m_s

$\hbar/2$	$-\hbar/2$
$\lvert 0>$	$\lvert 1>$
up	down



Bloch sphere

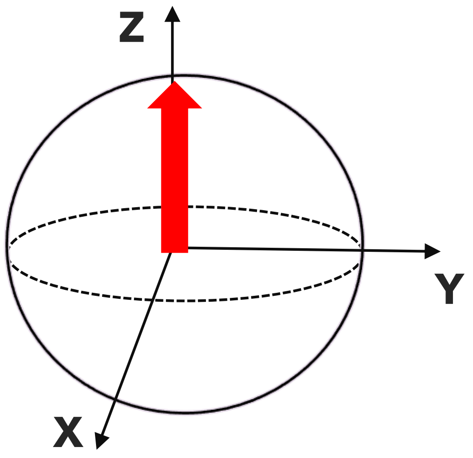
$$|\varphi> = \cos\left(\frac{\theta}{2}\right)|0> + e^{i\alpha}\sin\left(\frac{\theta}{2}\right)|1>$$

It seems very hard to explain what a quantum spin is, so can we say at least what it DOES?

$$R(\alpha,n)|\Psi> = e^{-i\alpha(S.n)/\hbar}|\Psi> \;\; = e^{-i\alpha(\sigma.n)}|\Psi>$$

$$S^2 = \hbar^2 s(s+1) \;\; = S_x^2 + S_y^2 + S_z^2$$

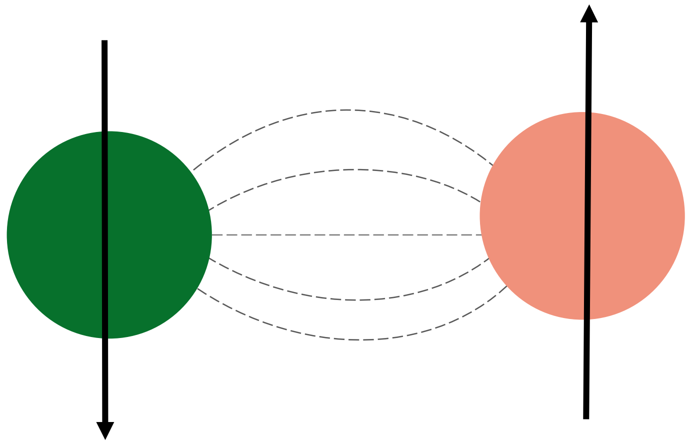
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin $s = 0$	Spin $s = \frac{1}{2}$
$S^2 = 0 \longrightarrow S_n = 0$	$R(\alpha,n) = \cos\left(\frac{\alpha}{2}\right) I - i(\sigma.n)\sin\left(\frac{\alpha}{2}\right)$
$R(\alpha,n)|\Psi> = |\Psi>, \quad \forall \;\; \alpha, n\forall$	$R(\alpha,X)\,|0> \; = \cos\left(\frac{\alpha}{2}\right)|0> -i\sin\left(\frac{\alpha}{2}\right)\;|1>$ 

II- Entanglement

Entanglement is a phenomenon in which a single wave function describes two separate quantum objects.

These quantum objects share the same existence, no matter how far apart they might be.



Entangled states can be composed of more than two qubits.

System of 2 qubits

$|\varphi >= |00 >= |0 > \otimes |0 >$ Separable state

$|\Psi >= \frac{1}{\sqrt{2}} (|00 > +|11 >)$ Entangled state

System of 3 qubits

$|GHZ >= \frac{1}{\sqrt{2}} (|000 > +|111 >)$ Maximally entangled state

$|W >= \frac{1}{\sqrt{3}} (|001 > +|010 > +|100 >)$ Entangled state

$= \frac{1}{\sqrt{3}} (|0 > \otimes (|01 > +|10 >) + |1 > \otimes |00 >)$

III- Expectation value of the spin

Expectation value of a discrete random variable in probability is given by

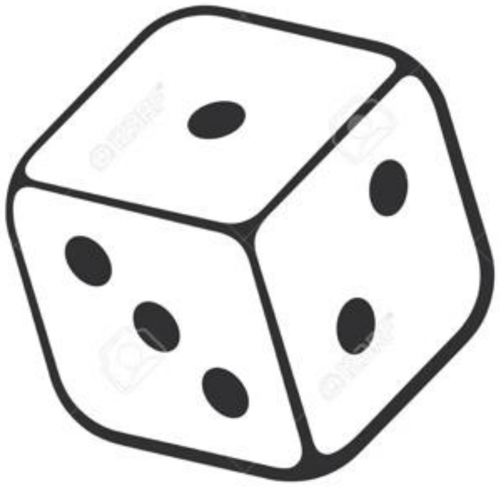
$$\mu = E(X) = \sum_x xp(x)$$

↓

This informs on the long-term results that one would obtain.

Roll a dice 4 times and record the number of one.

X = Random variable



x	0	1	2	3	4
$p(x)$	0.48	0.39	0.11	0.015	0.0007

The values $\boldsymbol{x}$ that $\boldsymbol{X}$ can take

The Expected value of the number of one in the 4 rolling of the dice is

$$\mu = 0.(0.48) + 1.(0.39) + 2.(0.11) + 3.(0.015) + 4.(0.0007) = 0.66$$

Expectation value in quantum mechanics

$$< A >_{\varphi} = < \varphi \mid A \mid \varphi >$$

↓

$$< A >_{\varphi} = \sum_{i} a_i | < \varphi | \Psi_i > |^2$$

Expectation value of a ½-spin system along a given direction **n**

$$|+n><+n| + |-n><-n| = 1$$

$$< S_n >_{\varphi} = < \varphi \,|\, S_n \,|\, \varphi > = \frac{\hbar}{2} < \varphi \,|\sigma_n\,|\, \varphi >$$

$$= \frac{\hbar}{2} < \varphi \,|\sigma_n\,| + n >< +n|\varphi > + \frac{\hbar}{2} < \varphi \,|\sigma_n\,| - n >< -n|\varphi >$$

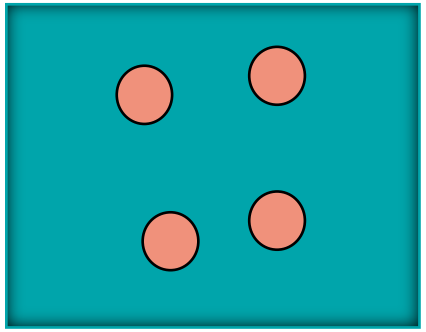
$$= \frac{\hbar}{2} < \varphi \,|+n >< +n|\varphi > - \frac{\hbar}{2} < \varphi \,| - n >< -n|\varphi >$$

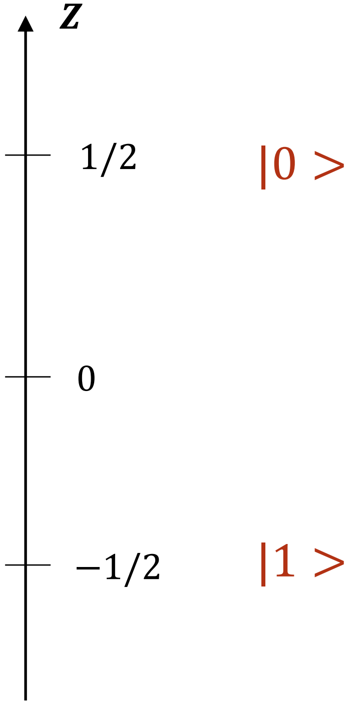
$$= \frac{\hbar}{2} \left| < \varphi \,| + n > \right|^2 - \frac{\hbar}{2} \left| < \varphi \,| - n > \right|^2$$

$$= \frac{\hbar}{2} P(+n) - \frac{\hbar}{2} P(-n)$$

Expectation value in single qubit states

$|\Psi> = |0>$

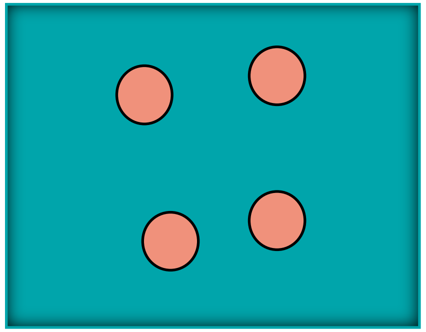


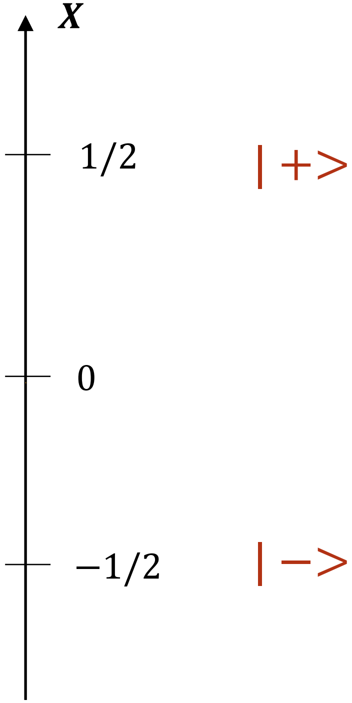


$\longrightarrow$ $< \Psi|\sigma_Z|\Psi > = 1$

Expectation value in single qubit states

$$|\Psi> = |0> = \frac{1}{\sqrt{2}}(|+> + |->)$$





$$\longrightarrow \quad <\Psi|\sigma_X|\Psi> = 0$$

Expectation value of the spin in entangled states

- **Separable two qubit states**

$$|\Psi> = |00> = |0> \otimes |0>$$

$$< \Psi|\, \sigma_Z \otimes I|\Psi> = 1$$

$$< \Psi|\, I \otimes \sigma_Z|\Psi> = 1$$

$$|\varphi> = \frac{1}{\sqrt{2}}(|00> + |01>) = |0> \otimes \underbrace{\frac{1}{\sqrt{2}}(|0> + |1>)}_{|+_x>}$$

$$< \Psi|\, \sigma_Z \otimes I| > = 1$$

$$< \Psi|\, I \otimes \sigma_Z|\Psi> = 0$$

$$< \Psi|\, I \otimes \sigma_X|\Psi> = 1$$

- **Entangled two qubit states**

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\langle\Psi|\,\sigma_{\mathrm{n}} \otimes I|\Psi\rangle = 0$$

$$\langle\Psi|\,I \otimes \sigma_n|\Psi\rangle = 0$$

n is any direction

There exists <u>no direction</u> that cancels out the lack of information

- **Entangled three qubit states**

$$|GHZ> = \frac{1}{\sqrt{2}}(|000> + |111>)$$

↓

A maximally entangled state

$$< GHZ|\, \sigma_n \otimes I \otimes I|GHZ> = 0$$

$$< GHZ|\, I \otimes \sigma_n \otimes I|GHZ> = 0$$

$$< GHZ|\, I \otimes I \otimes \sigma_n|GHZ> = 0$$

n is any direction

$$|W> = \frac{1}{\sqrt{3}}(|001> + |010> + |100>)$$

↓

An entangled state

$$< W|\, \sigma_z \otimes I \otimes I|W> = \frac{2}{3} < 1$$

$$< W|\, I \otimes \sigma_z \otimes I|W> = \frac{2}{3} < 1$$

$$< W|\, I \otimes I \otimes \sigma_z|W> = \frac{2}{3} < 1$$

Only along the **z** direction

IV- Entanglement measure

Entanglement measure: is a theoretical measure that allows to quantify the amount of entanglement in a quantum system.

Why is quantifying entanglement important?

1- Quantum computing

- Superdense coding
- Cryptography
- Quantum teleportation
- Machine learning

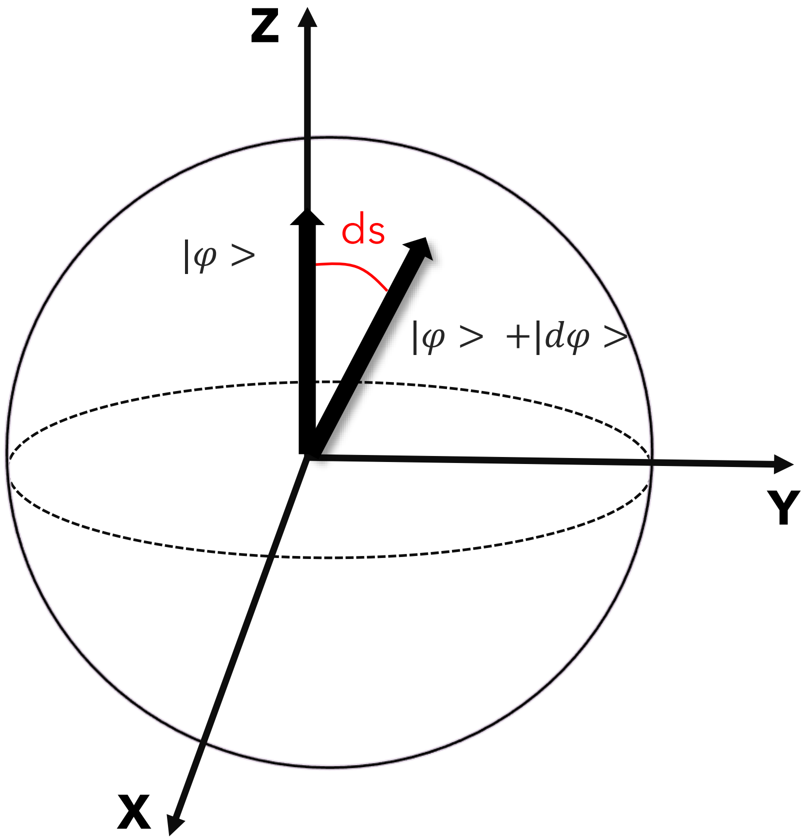
2- Quantum sensing

3- Quantum biology   arXiv preprint arXiv:1006.4053

Constructing the entanglement measure

Axioms

- Vanishing on separable states
- Invariant under local unitary operations
- Monotonicity under local operators and classical communications



Fubini-Study metric

$$ds^2 = < d\varphi|d\varphi > - < \varphi|d\varphi >< d\varphi|\varphi >$$

One qubit system

$|\varphi>$ ⟶ $R(\alpha, n)|\varphi>$

$|d\varphi>$ ⟶ $dR\ (R(\alpha, n)|\varphi>)$

$$g = \left(< \varphi|(\sigma.n)^2|\varphi > \ - < \varphi|(\sigma.n)|\varphi >^2\right) d\alpha^2 = (1- < \varphi|(\sigma.n)|\varphi >^2)\ d\alpha^2 = Var\left((\sigma.n)\right) d\alpha^2$$

$|0\rangle \longrightarrow g = (1 - \langle 0|(\sigma.n)|0\rangle^2)\, d\alpha^2 \longrightarrow E = 1 - \max_n \langle 0|(\sigma.n)|0\rangle^2 = 1 - \langle 0|(\sigma_z)|0\rangle^2 = 0$

$E = \inf_n \; (1 - \langle 0|(\sigma.n)|0\rangle^2)$

M qubit system

$|\varphi\rangle \longrightarrow \prod_{\mu=1}^{M} R(\alpha, n)_\mu |\varphi\rangle = \prod_{\mu=1}^{M} (e^{-i\alpha(\sigma.n)})_\mu |\varphi\rangle$

$|d\varphi\rangle \longrightarrow \sum_{\mu=1}^{M} dR_\mu \; |\varphi\rangle_R = \sum_\mu (-i\, d\alpha\, (\sigma.n))_\mu |\varphi\rangle_R$

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, d\alpha^{\mu} d\alpha^{\nu} = \sum_{\mu\nu} \left(< \varphi|(\sigma.n)_{\mu}(\sigma.n)_{\mu}|\varphi > - < \varphi|(\sigma.n)_{\mu}|\varphi >< \varphi|(\sigma.n)_{\nu}|\varphi > \right) d\alpha^{\mu} d\alpha^{\nu}$$

$$1 - < \varphi|(\sigma.n) \otimes I \ldots \otimes I|\varphi >^2$$

$$g = \begin{pmatrix} 1 - < \varphi|(\sigma.n)_1|\varphi >^2 & \cdots & g_{1M} \\ \vdots & \ddots & \vdots \\ g_{M1} & \cdots & 1 - < \varphi|(\sigma.n)_M|\varphi >^2 \end{pmatrix}$$

$$g = \begin{pmatrix} Var[(\sigma.n)_1] & \cdots & g_{1M} \\ \vdots & \ddots & \vdots \\ g_{M1} & \cdots & Var[(\sigma.n)_M] \end{pmatrix}$$

$$\mathrm{E} = \inf_{n} \left[Tr\,(g) \right] = \inf_{\mathrm{n}} \sum_{\mu} Var\,[\,(\sigma.n)_{\mu}]$$

V- Conclusions

- The spin is an elusive property, but it informs on how a given particle transforms under rotations.

- The expectation value of a spin along a given direction $< \boldsymbol{\psi} |\sigma_{\boldsymbol{n}}|\boldsymbol{\psi} >$, informs on the lack of information that the observer has on the spin of $|\psi >$ along $\boldsymbol{n}$.

- From a distance on the space of states, we derive an entanglement measure that is expressed as a function of the variance of the spin.

Thank you