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DIPARTIMENTO DI SCIENZE FISICHE, DELLA TERRA E DELL'AMBIENTE — DSFTA

Anderson Localization in Photonic Crystal Systems

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A photonic crystal is a material with a periodic variation of the refractive index, i.e. $n(\boldsymbol{r}) = n(\boldsymbol{r} + \boldsymbol{R})$

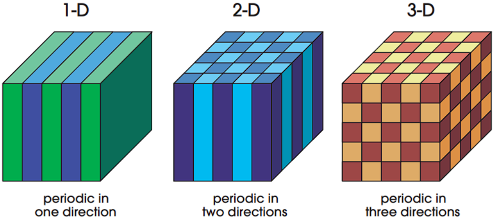

Figure: Reproduced from [1]

As a result, the propagation of light is affected: some modes can propagate, while others cannot

- Light propagation in medium is governed by *Maxwell equations in matter*, that is

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho, & \nabla \cdot \boldsymbol{B} = 0, \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, & \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}. \end{cases}$$

- Generally, the displacement field $\boldsymbol{D}$ may be written as a functional of the electric field $\boldsymbol{E}$, i.e. $\boldsymbol{D}[\boldsymbol{E}]$ (and similarly for $\boldsymbol{H}$ and $\boldsymbol{B}$)
- The equations that express this dependence are known as *constitutive relations*
- Quite generally, we can write

$$D_i/\varepsilon_0 = \sum_j \varepsilon_{ij} E_j + \sum_{j,k} \chi_{ijk} E_j E_k + O(E^3)$$

- For many dielectrics, we can make use of the following approximations
  1. Linear-response regime, i.e. we neglect terms $\propto E^2$ and higher
  2. Isotropy, that is, $\varepsilon_{ij} = \delta_{ij}\varepsilon$
  3. Non-dispersive material, i.e. $\varepsilon(\omega, \boldsymbol{r}) \equiv \varepsilon(\boldsymbol{r})$
  4. Transparent material, that is, $\varepsilon(\boldsymbol{r})$ is real and positive
- With these approximations and with no sources ($\rho = 0$, $\boldsymbol{J} = \boldsymbol{0}$), the equations become

$$\begin{cases} \nabla \cdot [\varepsilon(\boldsymbol{r})\boldsymbol{E}(t, \boldsymbol{r})] = 0, & \nabla \cdot \boldsymbol{H}(t, \boldsymbol{r}) = 0, \\ \nabla \times \boldsymbol{E}(t, \boldsymbol{r}) + \mu_0 \frac{\partial \boldsymbol{H}(t,\boldsymbol{r})}{\partial t} = \boldsymbol{0}, & \nabla \times \boldsymbol{H} - \varepsilon_0 \varepsilon(\boldsymbol{r}) \frac{\partial \boldsymbol{E}(t,\boldsymbol{r})}{\partial t} = \boldsymbol{0}. \end{cases}$$

- Since Maxwell equations are linear, we separate the time dependence from the spatial dependence by expanding into *harmonic modes* as

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r})e^{-i\omega t} \quad \text{and} \quad \boldsymbol{H}(t, \boldsymbol{r}) = \boldsymbol{H}(\boldsymbol{r})e^{-i\omega t}$$

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- The two divergence equations tell us that the field configurations are *transverse*
- The two curl equations can be decoupled and we end up with the *master equation*

$$\hat{\Theta}\boldsymbol{H}(\boldsymbol{r}) = \left(\frac{\omega}{c}\right)^2 \boldsymbol{H}(\boldsymbol{r})$$

where the linear and hermitian operator $\hat{\Theta}$ is defined as $\hat{\Theta}[\cdot] \doteq \nabla \times \left[\frac{1}{\varepsilon(\boldsymbol{r})}\nabla \times [\cdot]\right]$, and, for $\varepsilon > 0$ it is also *positive semi-definite*

- Then, we recover $\boldsymbol{E}$ as

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{i}{\omega\varepsilon_0\varepsilon(\boldsymbol{r})}\nabla \times \boldsymbol{H}(\boldsymbol{r}).$$

In this way $\varepsilon\boldsymbol{E}$ is transverse by construction

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- If the permittivity is periodic in space, e.g. $\varepsilon(\boldsymbol{r} + \boldsymbol{R}) = \varepsilon(\boldsymbol{r})$, then $[\hat{\Theta}, \hat{T}_{\boldsymbol{R}}] = 0$, and the solution can be written as

$$\boldsymbol{H}_{\boldsymbol{k}}(\boldsymbol{r}) = e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} u_{\boldsymbol{k}}(\boldsymbol{r})$$

  with $u_{\boldsymbol{k}}(\boldsymbol{r} + \boldsymbol{R}) = u_{\boldsymbol{k}}(\boldsymbol{r})$. This result is generally known as *Bloch Theorem*
- Since we can focus only on a finite region of the space, the eigenvalues are discretized by the band index n, i.e. $\omega \to \omega_n(\boldsymbol{k})$, where also $\boldsymbol{k}$ varies in a finite region known as *Brillouin zone*

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- Let us suppose $\boldsymbol{R} = ma\hat{\boldsymbol{y}}$ where m is an integer and a is the *lattice constant* (see the figure below)

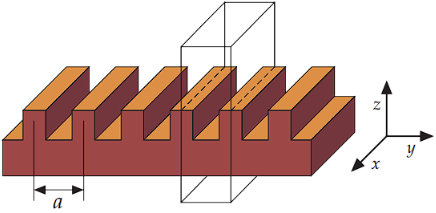

Figure: Reproduced from [1]

- Recall that, for a general rotation $\mathcal{R} \in O(3)$...
  1. the field $\boldsymbol{E}(\boldsymbol{r})$ transforms as $\boldsymbol{E}'(\boldsymbol{r}) \equiv \hat{\mathcal{O}}_{\mathcal{R}}\boldsymbol{E}(\boldsymbol{r}) = \mathcal{R}\boldsymbol{E}(\mathcal{R}^{-1}\boldsymbol{r})$
  2. the field $\boldsymbol{H}(\boldsymbol{r})$ transforms as $\boldsymbol{H}'(\boldsymbol{r}) \equiv \hat{\mathcal{O}}_{\mathcal{R}}\boldsymbol{H}(\boldsymbol{r}) = \det(\mathcal{R})\mathcal{R}\boldsymbol{H}(\mathcal{R}^{-1}\boldsymbol{r})$

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- In the system above, there is mirror symmetry in the *yz* plane, expressed by the reflection matrix $I_x = \mathrm{diag}(-1, 1, 1) \in O(3)$
- Since $\det(I_x) = -1$, and $\hat{\mathcal{O}}_{I_x} \boldsymbol{F} = \lambda_\pm \boldsymbol{F}$ with $\lambda_\pm = \pm 1$ and $\boldsymbol{F} = \{\boldsymbol{H}, \boldsymbol{E}\}$, we have
  1. for $\lambda_+ \to$ *TM modes* $\to$ field configuration $\{H_x, E_y, E_z\}$
  2. for $\lambda_- \to$ *TE modes* $\to$ field configuration $\{E_x, H_y, H_z\}$
- Anyway...since ε depends only on y, there is *continuous* invariance along x and z, that is, $\boldsymbol{H}$ and $\boldsymbol{E}$ are eigenfunctions of the translation operators $\hat{T}_x$ and $\hat{T}_z$
- As a consequence, the field can be written as

$$\boldsymbol{F}(x, y, z) = e^{-i\boldsymbol{k}_{||}\cdot(x,0,z)}\tilde{\boldsymbol{F}}(y)$$

with $\boldsymbol{k}_{||} = (k_x, 0, k_z)$

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- So what? We can always rotate the system such that $\boldsymbol{k}_{||} = (k_x, 0, k_z) \to (0, 0, k_z')$, i.e. $k_x = 0$
- The consequence of this is that the field configurations in the *TE* or *TM* modes can always be constructed, not only in the *yz* plane
- In the *TM* modes, the differential equation for $\tilde{H}_x(y)$ is

$$-\frac{d}{dy}\left(\frac{1}{\varepsilon(y)}\frac{d\tilde{H}_x(y)}{dy}\right) = \left(\omega^2 - k_{||}^2/\varepsilon(y)\right)\tilde{H}_x(y)$$

- Let us consider a simulation in which we have two materials with lattice constant $a = d_A + d_B$, with d_i the thickness of material $i = A, B$, and, for example, $n_A = 1.5$ and $n_B = 3.5$
- For simplicity, we put $k_{||} = 0$. As a consequence, from the equations we have $E_y = 0$

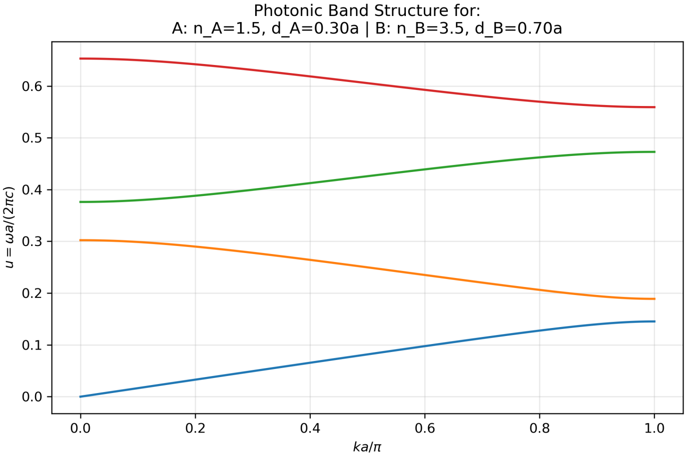

Figure: First 4 bands with $\varepsilon(y) = n_A^2/\varepsilon_0$ for $0 < y < d_A$ and $\varepsilon(y) = n_B^2/\varepsilon_0$ for $d_A < y < d_A + d_B$

- So far, we have considered perfect objects, that is, they can be built such that the function $\varepsilon(\boldsymbol{r})$ is periodic
- For real systems, we must take into account also possible *perturbations/defects/disorders* of the variables...
- The *Anderson localization* is the localization of a wave due to the degree of *randomness* of the system
- For example, we may consider something like $\varepsilon_{real}(\boldsymbol{r}) = \varepsilon_{ideal}(\boldsymbol{r}) + \delta\varepsilon(\boldsymbol{r})$ with $\delta\varepsilon(\boldsymbol{r})$ distributed with some probability distribution function

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- To get an idea of what Anderson localization is, let us consider the following Hamiltonian

$$\hat{H} = -J \sum_n \left(\hat{c}^\dagger_{n+1} \hat{c}_n + \hat{c}^\dagger_n \hat{c}_{n+1} \right) + \sum_n \epsilon_n \hat{c}^\dagger_n \hat{c}_n$$

  and the evolution of a *single* particle starting at $|\psi(0)\rangle = |n_0\rangle$
- The first term encodes the tunnelling between adjacent sites while the second term represents the on-site energies
- In absence of disorder (as the case of a lattice with $\epsilon_n \equiv \epsilon_0$ or ϵ_n periodic), the state, in the simple case $\epsilon_n = \epsilon_0$, evolves as

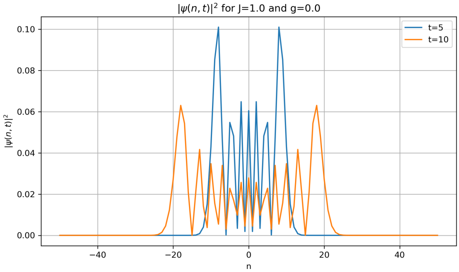
$$|\psi(t)\rangle = \frac{1}{\sqrt{N}} \sum_k e^{-ikn_0} e^{-iE(k)t} |k\rangle,$$

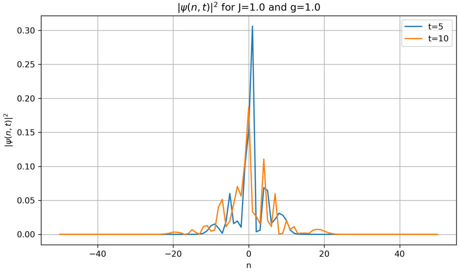
  i.e., it is *delocalized*. Moreover, the energies are $E(k) = -2J\cos(k)$

Simulation: *Anderson-Tight-Binding.ipynb*

- To simulate disorder, we can generate random energies ϵ_n and solve the eigenvalue problem
- In the code, the energies are generated as $\epsilon_n \sim \mathcal{N}(0, g^2)$ for $n = -N, \dots, N$
- Concretely, this model simulates a 1D lattice with impurities...
- The code is open and free to use in the indico page :D

Simulation: *Anderson-Tight-Binding.ipynb*





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- It should be clear that there are similarities between the electrodynamics equations written so far and the time-independent Schrodinger equation, that is

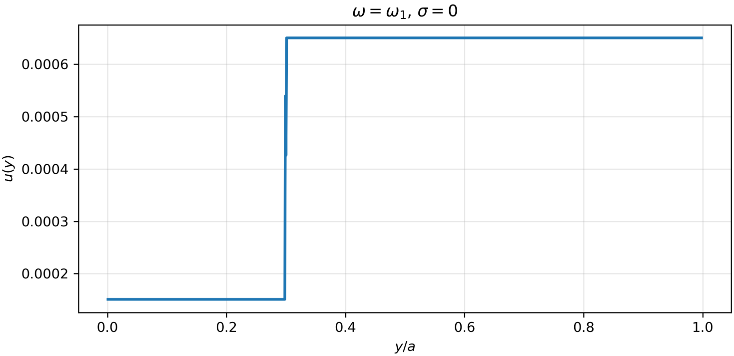
$$\hat{\Theta}\boldsymbol{H}(\boldsymbol{r}) = \left(\frac{\omega}{c}\right)^2 \boldsymbol{H}(\boldsymbol{r}), \quad \hat{H}\Psi = E\Psi$$

- In this case, we can simulate disorder by introducing a random uncertainty in the permittivity, that is $\varepsilon_{real}(\boldsymbol{r}) = \varepsilon_{ideal}(\boldsymbol{r}) + \delta\varepsilon(\boldsymbol{r})$
- To check if the fields localize, we can compute the energy density

$$u(y) = \frac{1}{4}\left(|\varepsilon(y)E_z(y)|^2 + |\mu_0 H_x(y)|^2\right)$$

- Here, there is no periodicity. For a correct physical sense it is better to use *Dirichlet boundary conditions*, that is, we imagine the fields in a *cavity*

- Technically, there is no point in talking about wavenumbers k_y...we only have a discrete set of frequencies $\omega_1, \omega_2, \ldots$
- We generate the noise as $\delta\varepsilon(y) \sim \mathcal{N}(0, \sigma^2)$ and compute the effective permittivity as $\varepsilon_{real}(y) = \varepsilon_{ideal}(y)(1 + \sigma\delta\varepsilon(y))$
- In absence of disorder we should see delocalized fields...


$\omega = \omega_1$, $\sigma = 0$
0.0006
0.0005
0.0004
0.0003
0.0002
$u(y)$
0.0
0.2
0.4
0.6
0.8
1.0
y/a

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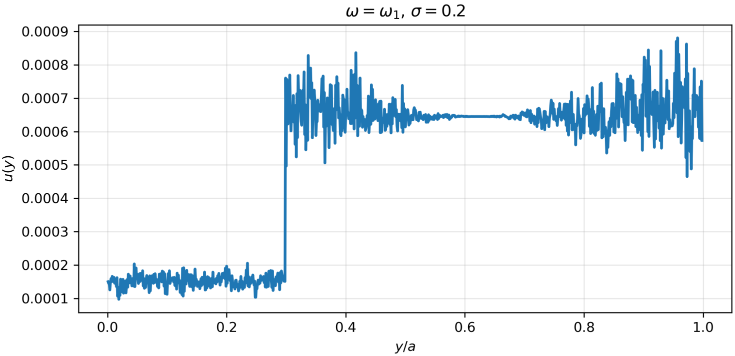
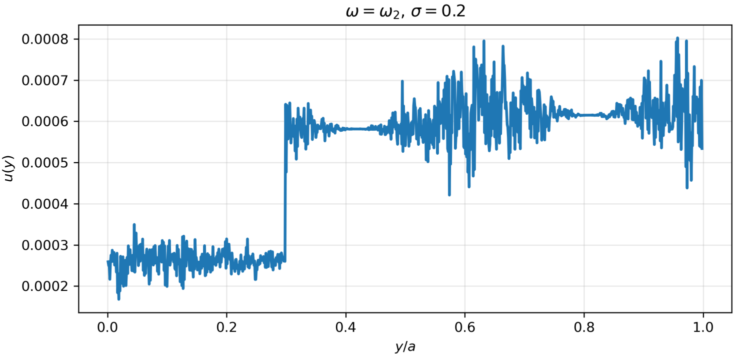
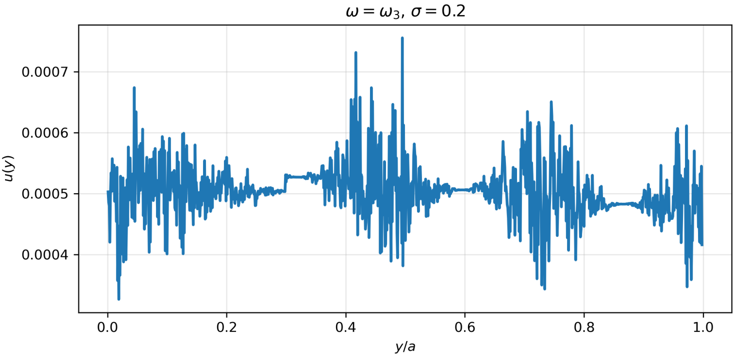



$\omega = \omega_2$, $\sigma = 0.2$
0.0008
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
$u(y)$
0.0
0.2
0.4
0.6
0.8
1.0
y/a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$u(y)$
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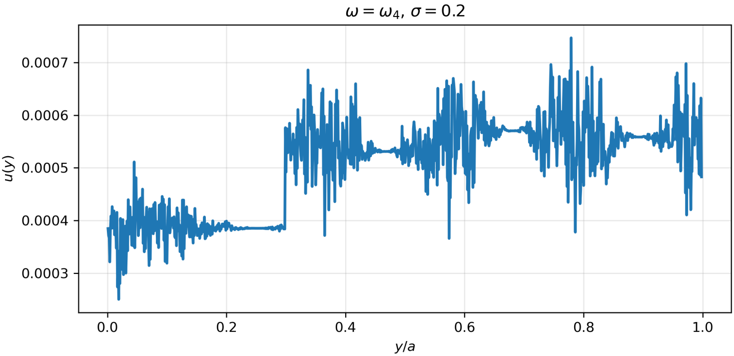
0.0006
0.0005
0.0004
0.0
0.2
0.4
0.6
0.8
1.0
y/a


$\omega = \omega_4$, $\sigma = 0.2$
0.0007
0.0006
0.0005
0.0004
0.0003
$u(y)$
0.0
0.2
0.4
0.6
0.8
1.0
y/a

Conclusions and Outlook

- A photonic crystal is a material with a periodic variation of the refractive index
- Symmetries help us simplify physical problems by reducing their complexity and revealing what really determines the system
- Anderson localization is a key effect that strongly influences light confinement in disordered photonic structures
- Future work would aim to explore higher-dimensional disordered structures and their possible applications in photonic devices
- Maybe, when there's some free time, we could try to simulate the Anderson effect in more complex topologies...

Thank you! :D

References I

[1] John D. Joannopoulos et al. *Photonic Crystals: Molding the Flow of Light - Second Edition*. REV - Revised, 2. Princeton University Press, 2008. ISBN: 9780691124568. URL: http://www.jstor.org/stable/j.ctvcm4gz9 (visited on 10/18/2025).