Stellar Physics

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Overview

Topics:

- Hydrostatic equilibrium
- T_C and P_C
- Virial Theorem
- Kelvin-Helmholtz time scale
- Nuclear reactions time scale
- Nuclear fusion: the classical viewpoint
- Nuclear fusion: the quantum viewpoint
- Nuclear fusion: an in-depth analysis (cross section, rate, Gamow peak, …)

Useful values:

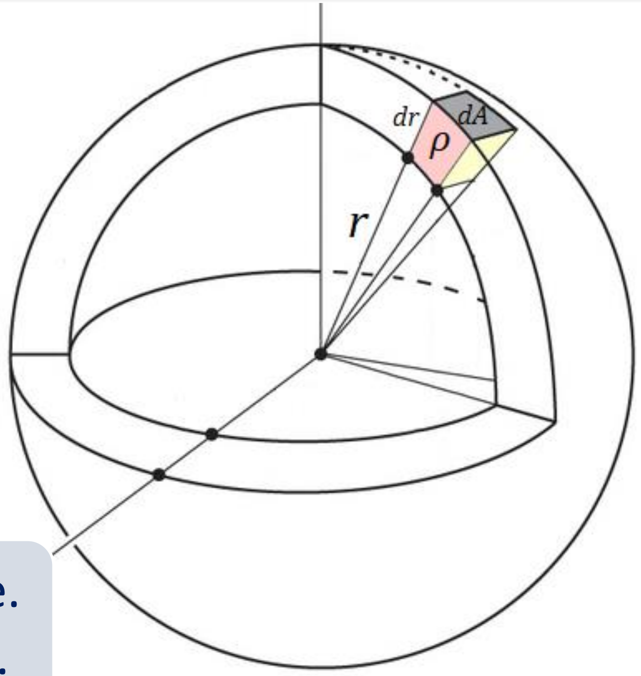
- $G = 6{,}67 \cdot 10^{-8}\, cm^3/s^2 g$
- $M_\odot = 1{,}99 \cdot 10^{33} g$
- $R_\odot = 6.96 \cdot 10^{10} cm$
- $L_\odot = 3{,}827 \cdot 10^{26}\, J/s$
- $k_B = 1{,}38 \cdot 10^{-23}\, J/K$
- $m_H = 1{,}67 \cdot 10^{-24} g$
- $\hbar = 1{,}05 \cdot 10^{-34} J\, s$
- $\varepsilon_0 = 8{,}85 \cdot 10^{-12}\, C^2/N\, m^2$
- $e = 1.6 \cdot 10^{-19} C$
- $1eV = 1.6 \cdot 10^{-19} J$

Hydrostatic equilibrium

Observational evidence

Characteristic of the sun:
- Self-gravitating system
- Spherical shape
- Constant dimensions

Is it in **equilibrium?**



Free fall time $\tau_{ff\odot}$

Consider an infinitesimal volume at $r = R_\odot$ subject only to the gravitational force.
The $\tau_{ff\odot}$ can be evaluated from the uniformly accelerated motion $x(t) = gt^2/2$.
Using $g_\odot = GM_\odot/R_\odot^2$

$$\tau_{ff\odot} = \sqrt{\frac{2R_\odot}{g_\odot}} = \sqrt{\frac{2R_\odot^3}{GM_\odot}} \approx 35\ min \qquad \left(\boldsymbol{\tau_{ff\odot} \ll \Delta t_{star\ evolution}}\right)$$

→ **Hydrostatic Equilibrium**

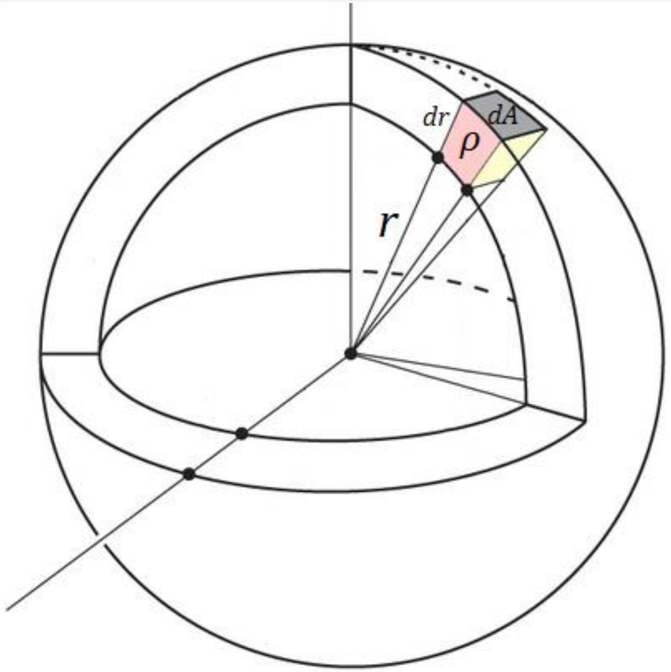
Similar result for:
- Red Giant: $M \sim M_\odot,\ R \sim 10^2 R_\odot$ so $\tau_{ff} \approx 24\ d$
- White Dwarf: $M \sim M_\odot,\ R \sim 10^{-2} R_\odot$ so $\tau_{ff} \approx 2{,}1\ s$

Hydrostatic equilibrium

Assumptions

We will assume the stars to be:

- Self-gravitating systems
- Spherically symmetric
- Gaseous
- Non-rotating
- Without strong magnetic fields



Proof

Under the assumptions, the only forces acting on a mass element $dm = \rho\, dr\, dA$ come from **pressure** and **gravity**

$$F_P(r) = F_g(r)$$

$$P\, dA + [-(P + dP)\, dA] = \frac{GM(r)}{r^2} \rho\, dr\, dA$$

$$\boldsymbol{\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho}$$

This is the **Hydrostatic Equilibrium** equation

P_C and T_C

We can use it to estimate:

- Central pressure P_C
- Central temperature T_C

P_C and T_C

P_C

Let's **replace the quantities** in the Hydrostatic Equilibrium equation **with averages**

- $\frac{dP}{dr} \to \frac{(P_C - 0)/2}{(0 - R_\odot)/2} = -\frac{P_C}{R_\odot}$
- $-\frac{GM(r)}{r^2}\rho \to -\frac{G\,(M_\odot - 0)/2}{[(R_\odot - 0)/2]^2}\left(\frac{M_\odot}{{}^4/_3\,\pi R_\odot^3}\right) = -\frac{3}{2\pi}\frac{GM_\odot^2}{R_\odot^5}$

Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho$$

By equating we get

$$P_C = \frac{3}{2\pi}\frac{GM_\odot^2}{R_\odot^4} \approx 5{,}3 \cdot 10^9 atm$$

T_C

To relate P_C to T_C we need the unknown **plasma state equation**. So, we will assume the plasma to be:

- **Ideal gas** $P = nk_BT$, where n is the particle numerical density $n = N/V_\odot$
- **Hydrogen** contributing 2 particles per atom $N = 2\,M_\odot/m_H$

$$T_C = \frac{P_C}{nk_B} = \frac{P_C}{k_B}\left(\frac{2\,M_\odot/m_H}{{}^4/_3\,\pi R_\odot^3}\right)^{-1} \approx 2{,}3 \cdot 10^7 K$$

The energy conservation and the virial theorem

Energy conservation

The **evolution of a star** can be inferred by looking at its **energy conservation** through

- **Total energy** equation
- **Virial theorem**

$$\begin{cases} E_{tot} = E_G + E_{in} \\ E_G = -3E_{in}(\gamma - 1) \end{cases}$$

Proof

$$\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

$$\int_0^R \frac{dP(r)}{dr} 4\pi r^3 dr = \int_0^R -\frac{GM(r)}{r^2}\rho(r) 4\pi r^3 dr$$

$$[P(r)4\pi r^3]_0^R - 3\int_0^R P(r)4\pi r^2 dr = \int_0^R -\frac{GM(r)}{r}\rho(r)4\pi r^2 dr$$

$$0 - 3\int_0^R \varepsilon_{in}(\gamma - 1)dV = \int_0^R -\frac{GM(r)}{r}\rho(r)dV$$

$$\boldsymbol{-3E_{in}(\gamma - 1) = E_G}$$

- Hydrostatic equilibrium
- $\int_0^R (\cdots)4\pi r^3 dr$
- Integrate by part

- $P(R) = 0 \quad \Rightarrow \quad [P(r)4\pi r^3]_0^R = 0$
- $dV = 4\pi r^2 dr$
- $P(r) = \varepsilon_{in}(\gamma - 1)$

- **Virial Theorem**

The energy conservation and the virial theorem

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$$\begin{cases} E_{tot} = E_{in}(-3\gamma + 4) \\ E_G = -3E_{in}(\gamma - 1) \end{cases}$$

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Stability condition

A **star is stable if** $E_{tot} < 0$. Since $E_{in} > 0$, then $\boldsymbol{\gamma > 4/3}$

- $E_{tot} < 0$ because energy is needed to take the star apart
- If $\gamma \leq 4/3$, the star is unstable
- For a **monoatomic gas** $\boldsymbol{\gamma = 5/3}$

The energy conservation and the virial theorem

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The **evolution of a star** can be inferred by looking at its **energy conservation** through

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$$\text{if } \gamma = \frac{5}{3} \Rightarrow \begin{cases} E_{tot} = -E_{in} \\ E_G = -2E_{in} \end{cases}$$

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Luminosity

The luminosity is the **energy radiated** out, so $\boldsymbol{L = -\dot{E}_{tot} > 0}$

$$\begin{cases} L = \dot{E}_{in} \\ \dot{E}_{in} = -\frac{1}{2}\dot{E}_G \end{cases} \Rightarrow \begin{cases} \dot{E}_{in} > 0 \\ \dot{E}_G < 0 \end{cases} \Rightarrow \begin{cases} \dot{T} > 0 \\ \dot{R} < 0 \end{cases}$$

A star gets hotter while contracting, it has a **negative specific heat**

Kelvin–Helmholtz time scale

$\dot{E}_G$ conversion

From the energy conservation we saw that

$$\begin{cases} L = \dot{E}_{in} \\ \dot{E}_{in} = -\frac{1}{2}\dot{E}_G \end{cases} \Rightarrow \begin{cases} L = -\frac{1}{2}\dot{E}_G \\ \dot{E}_{in} = -\frac{1}{2}\dot{E}_G \end{cases} \Rightarrow \begin{matrix} \frac{1}{2}\Delta E_G \text{ is radiated out} \\ \frac{1}{2}\Delta E_G \text{ becomes thermal } E \end{matrix}$$

The luminosity can be used to estimate the **age of the sun**

τ_{KH} time scale

Assuming that:

- **Constant luminosity**
- **Gravity** is the **only energy source**

Averaging over all the quantities

$$\tau_{KH} = \frac{{}^{1}\!/_{2}\,\Delta E_G}{L_\odot} = \frac{1}{2L_\odot}\frac{G(M_\odot/2)}{R_\odot/2}M_\odot \approx 1{,}6 \cdot 10^7 y \qquad (\boldsymbol{\tau_{KH} \ll \Delta t_{earth}})$$

Inconsistent with geological evidence suggesting $\Delta t_{earth} \approx 4{,}5 \cdot 10^9 y$

New Energy Source ?

Nuclear reactions time scale

Binding energy

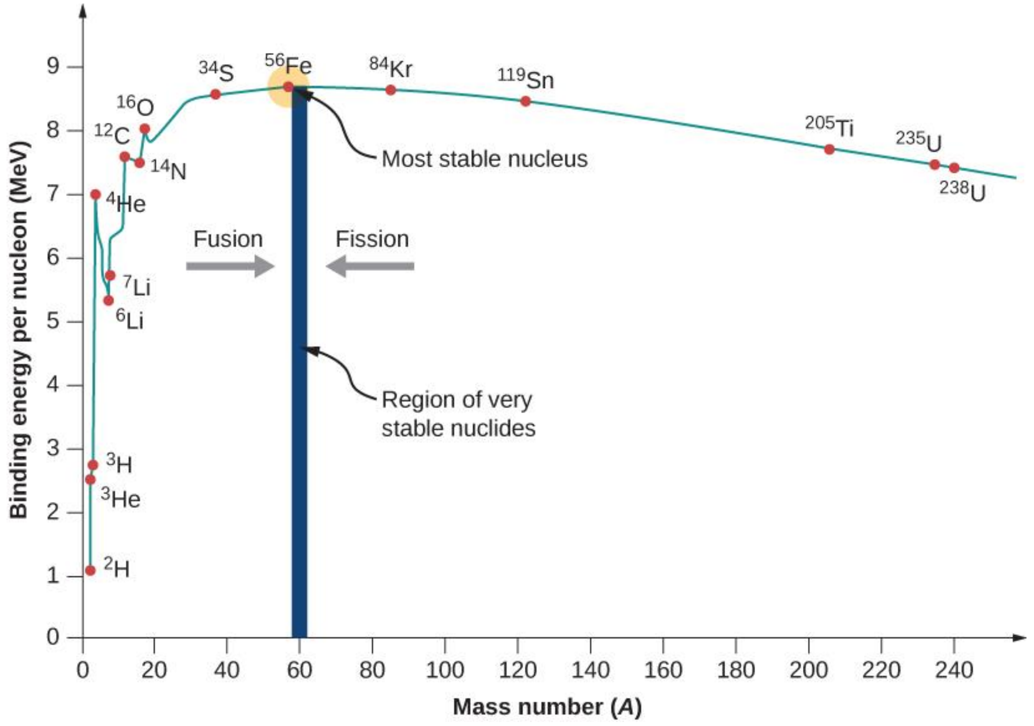
Considering a nucleus with

- A = atomic mass = $N_n + N_p$
- Z = atomic number = N_p

It is known that $\boldsymbol{m_{nucleus} < Zm_p + (A-Z)m_n}$.

This difference averaged over A is the **binding energy**

$$f = \frac{E_B}{A} = \frac{1}{A}\left[Zm_p + (A-Z)m_n - m_{nucleus}\right]c^2$$



τ_{nucl} time scale

If in the sun all the $\boldsymbol{p} \rightarrow \boldsymbol{He}$ and assuming

- Constant luminosity
- $f_{He} = 6{,}6MeV \approx 0.7\%\, m_n$

$$\tau_{nucl} = \frac{E_{nucl}}{L_\odot} = \frac{0{,}007 M_\odot c^2}{L_\odot} \approx 1{,}0 \cdot 10^{11} y$$

Since not all the p will be converted, this is consistent with $\boldsymbol{\Delta t_{universe} \approx 1{,}3 \cdot 10^{10} y}$

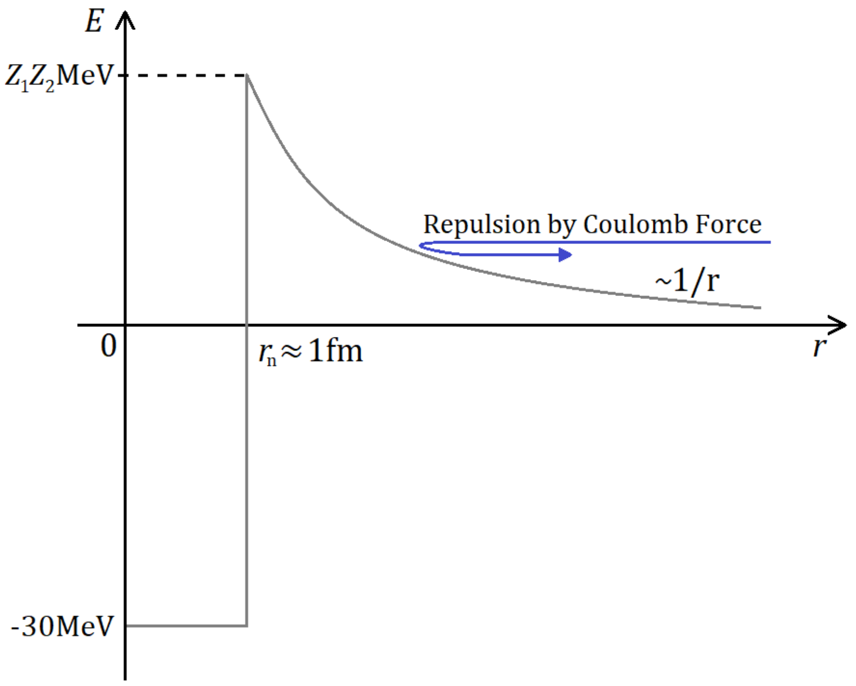
New Energy Source !

Nuclear fusion: the classical viewpoint

Coulomb barrier

Nuclear reactions requires the two nuclei to be at $\mathbf{r \approx 1fm}$.
The energy required to overcome the **Coulomb repulsion** is

$$E_{Coulomb} = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{r} \approx Z_1 Z_2 MeV$$



Thermal Energy

Assuming $T = T_C \approx 10^7 K$, the **average particle kinetic** energy is

$$E_T = k_B T \approx 1keV \qquad (E_T \ll E_{Coulomb})$$

Fusion probability

Classically, the **probability of a pp nuclear fusion** is

$$P_{pp} \propto \exp\left[-\frac{E_{Coulomb}}{k_B T}\right] = \exp[-10^3] \approx 10^{-434}$$

This is **too low** for $N_\odot \approx 10^{57}$ or even $N_U \approx 10^{80}$

Fusion is impossible!

(Classically)

Nuclear fusion: the quantum viewpoint

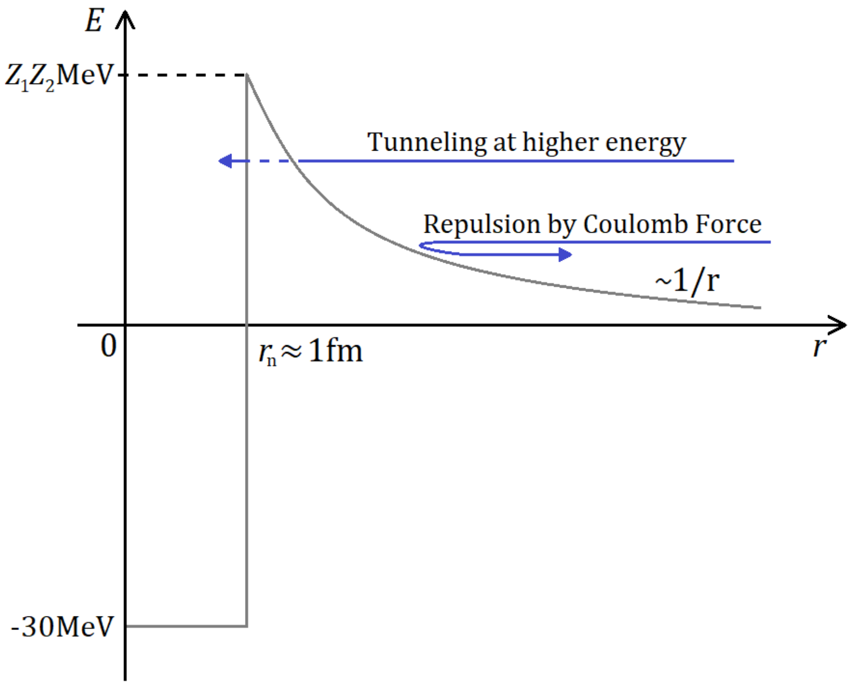
Quantum tunneling

Quantistically, a particle can overcome the Coulomb barrier through the **tunnel effect**. Its probability is

$$P \propto \exp\left[-\frac{b}{\sqrt{E}}\right] \quad \text{where} \quad b = \frac{1}{2\varepsilon_0 \hbar}\sqrt{\frac{m_r}{2}} Z_1 Z_2 e^2$$

- $m_r = m_1 m_2/(m_1 + m_2)$ is the reduced mass

For $\boldsymbol{Z_1 Z_2 = 1, T = 10^7 K}$ we have $\boldsymbol{P = 10^{-20}}$



Note:

- P is small ⇒ stars do not "explode"
- P is compatible with $N_\odot \approx 10^{57}$
- $Z_1 Z_2$ is at the exponent ⇒ cannot fuse nuclei with different Z at the same time

Fusion is possible!

(Quantistically)

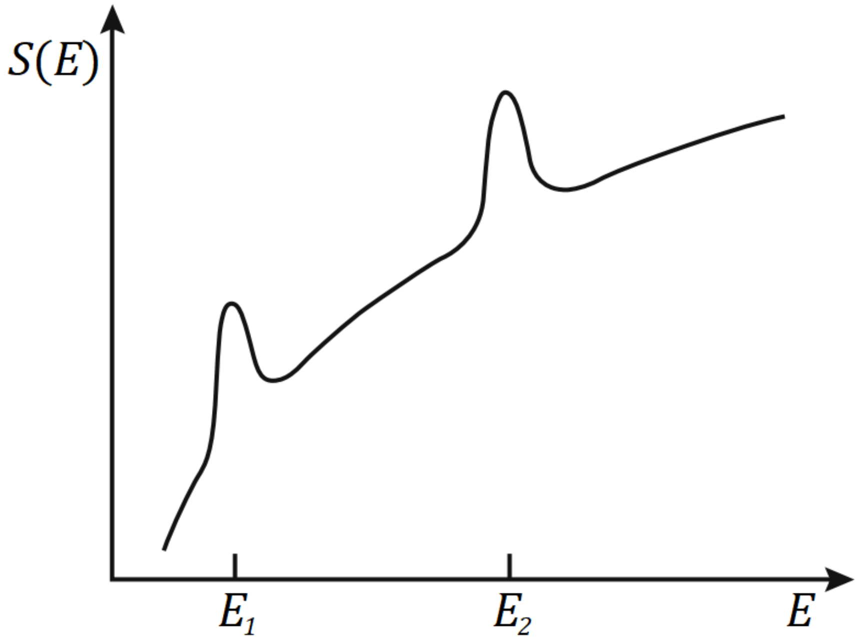
Nuclear fusion: the cross section

$\sigma(E)$

Using the probability P we can write the reaction cross section in terms of the energy of the colliding nuclei

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-\frac{b}{\sqrt{E}}\right] \quad \text{where} \quad b = \frac{1}{2\varepsilon_0 \hbar}\sqrt{\frac{m_r}{2}} Z_1 Z_2 e^2$$

- $E = m_r v^2/2$
- $m_r = m_1 m_2/(m_1 + m_2)$ is the reduced mass
- $\boldsymbol{S(E)}$ is the **astrophysical factor**



$S(E)$

The astrophysical factor

- Contains all **intrinsic nuclear properties** of the reaction
- Difficult to calculate or measure
- Shows resonances
- Measured in underground experiments (ex. INFN Gran Sasso)

Nuclear fusion: rate and Gamow Peak

Rate

The reaction rate can be written as

$$R_{12} = n_1 n_2 \langle \sigma v \rangle = n_1 n_2 \int_0^\infty \sigma(E) v f(E) dE$$

Where:

- $n_i = N_i / V$ is the numerical density
- v is the relative velocity of the nuclei
- $E = m_r v^2 / 2$
- $m_r = m_1 m_2 / (m_1 + m_2)$ is the reduced mass

Lifetime

From R_{12} we can estimate the lifetime of the sun

$$\Delta t_\odot = \frac{1}{n_p \langle \sigma v \rangle_{pp}} \approx 3 \cdot 10^9 y$$

Where we have used

- $n_p \approx 10^{26} \, 1/cm^3$
- $\langle \sigma v \rangle_{pp} \approx 10^{-43} \, cm^3/s$

$\sigma(E)$

The cross section is

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-\frac{1}{2\varepsilon_0 \hbar} \sqrt{\frac{m_r}{2}} \frac{Z_1 Z_2 e^2}{\sqrt{E}} \right]$$

$f(E)dE$

The Maxwell–Boltzmann distribution is

$$f(E)dE = \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{(k_B T)^{\frac{3}{2}}} \exp\left[-\frac{E}{k_B T} \right] dE$$

Nuclear fusion: rate and Gamow peak

Rate

The reaction rate can be written as

$$R_{12} = n_1 n_2 \frac{1}{\sqrt{\pi m_r}} \left(\frac{2}{k_B T}\right)^{\frac{3}{2}} \int_0^\infty S(E) \, \mathbf{exp}\left[-\frac{\boldsymbol{E}}{\boldsymbol{k_B T}}\right] \mathbf{exp}\left[-\frac{\boldsymbol{b}}{\sqrt{\boldsymbol{E}}}\right] dE$$

Where:

- $n_i = N_i/V$ is the numerical density
- v is the relative velocity of the nuclei
- $E = m_r v^2/2$
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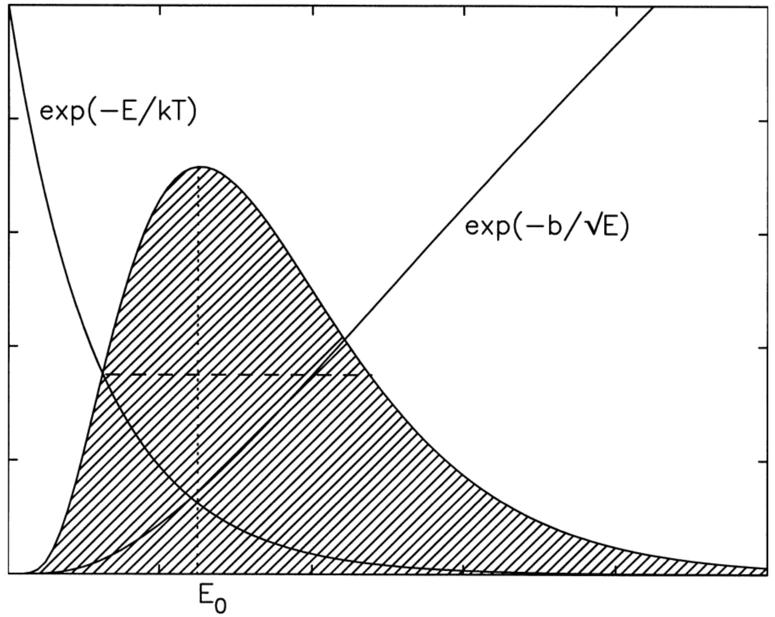


Gamow Peak

The energy at which the peak occurs is

$$E_0 = \left(\frac{1}{2} b k_B T\right)^{\frac{2}{3}} \quad \text{where} \quad b = \frac{1}{2\varepsilon_0 \hbar} \sqrt{\frac{m_r}{2}} Z_1 Z_2 e^2$$

which shows a strong dependence on the charge

Example

Consider the two reactions

- ${}^1H + {}^1H \quad \rightarrow \quad E_0 \approx 4.6 k_B T$
- ${}^{12}C + {}^{12}C \quad \rightarrow \quad E_0 \approx 111{,}3 k_B T$

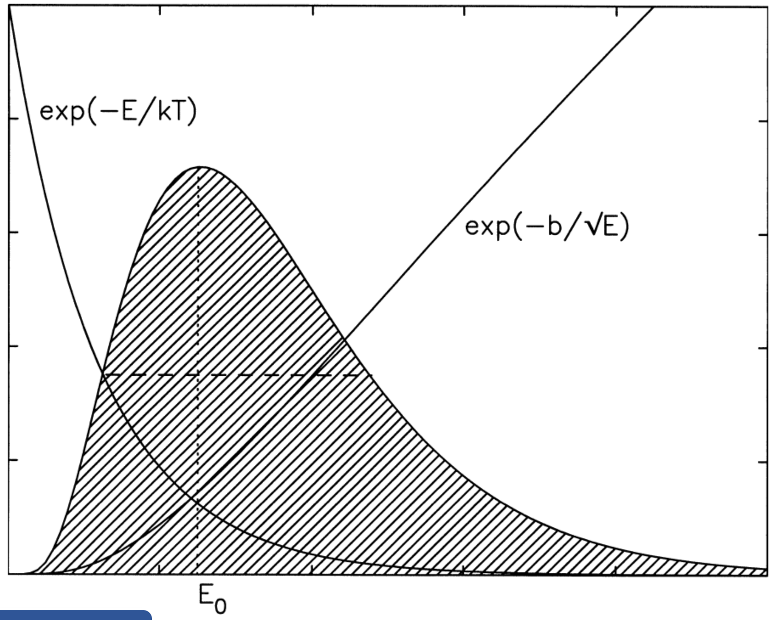
Nuclear fusion: rate and temperature

Rate

The reaction rate can be written in a simpler form by assuming:

- $S(E)$ does not have peaks (**non-resonant case**)
- $S(E)$ does not vary much around E_0, so $\boldsymbol{S(E) \approx S(E_0)}$
- The **Gamow peak** can be approximated by a **parabola**

$$R_{12} \propto \tau^2 e^{-\tau} \quad \text{where} \quad \tau = \frac{3E_0}{k_B T} \quad \text{and} \quad E_0 = \left(\frac{1}{2} b k_B T\right)^{\frac{2}{3}}$$



Temperature

The reaction rate can be rewritten as a **power low** to show it relates to the **temperature**

$$R_{12} \propto \tau^2 e^{-\tau} \quad \rightarrow \quad R \propto T^{\nu} \quad \text{where} \quad \nu = -\frac{2}{3} + \frac{\tau}{3}$$

the value of $\boldsymbol{\nu}$ **varies** strongly depending on $\boldsymbol{Z_1 Z_2}$

Example

- $^1H + {}^1H \quad \rightarrow \quad \nu \approx 5$
- $^1H + {}^{14}N \quad \rightarrow \quad \nu \approx 20$
- $^2He + {}^{12}C \quad \rightarrow \quad \nu \approx 42$
- $^{16}O + {}^{16}O \quad \rightarrow \quad \nu \approx 182$

Thank you for your attention