Polytropic Models

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An Introduction to Stellar Astrophysics: Prof. Pier Giorgio Prada Moroni

Guidelines

- The Equations of Stellar Evolution
    - Basic Assumptions
    - Full Set of Equations
    - The Role of Time
- Polytropic Models
    - Lane-Emden Equation
    - General Properties of the Solutions
    - Eddington Standard Model
    - Mass-Luminosity relation

The Equations of Stellar Evolution

- Isolation in space: their structure depends only on intrinsic properties (mass and composition)
- Spherical symmetry:

$$dm(t, r) = 4\pi r^2 \rho \, dr - 4\pi r^2 \rho \, v \, dt$$

Departure from spherical symmetry can arise from rotations and magnetic fields, but typically (e.g. the Sun)

$$\begin{cases} \frac{E_{rot}}{E_{grav}} = \frac{M\omega^2 R^2}{GM^2/R} \sim 2 \cdot 10^{-5} \\ \frac{U_{magn}}{E_{grav}/V} = \frac{B^2/\mu_0}{GM^2/R^4} \sim 10^{-10} \ \text{(for } B = 1\,T) \end{cases}$$

Full Set of Equations

- The stellar structure is determined by solving simultaneously the equations

$$\begin{cases} \partial P/\partial r = & \text{(Hydrodynamics equation)} \\ \partial m/\partial r = & \text{(Continuity equation)} \\ \partial L/\partial r = & \text{(Energy conservation)} \\ \partial T/\partial r = & \text{(Transport equation)} \\ \frac{\partial X_{i=1,N}}{\partial t} = & \text{(Equations for the composition)} \end{cases}$$

- Together with finite equations (equation of state, opacity, energy production, etc.)
- $4 + N$ partial differential equations for $4 + N$ unknown variables (m, ρ, T, L and X_i) in functions of *independent* variables t and r

The role of time

- Time appears in
  - Hydrodynamics equation $\rightarrow \partial^2 r/\partial t^2 \rightarrow$ changes in the mechanical structure $\rightarrow$ timescale

$$\tau_{dyn} \approx 10^3 \sqrt{(R/R_\odot)^3 (M_\odot/M)} s$$

  - Energy conservation $\rightarrow \partial s/\partial t \rightarrow$ changes in the thermal structure $\rightarrow$ timescale

$$\tau_{KH} \approx 10^{14} (M/M_\odot)^2 (R_\odot/R)(L_\odot/L) s$$

  - Equations for the composition $\rightarrow \partial X_i/\partial t \rightarrow$ changes in the composition $\rightarrow$ timescale

$$\tau_{nuc} \approx 10^{17} (M/M_\odot)(L_\odot/L) s \quad \text{(e.g. hydrogen burning)}$$

$\longrightarrow \tau_{nuc} \gg \tau_{KH} \gg \tau_{dyn}$

Complete Equilibrium

- Since $\tau_{nuc} \gg \tau_{KH} \gg \tau_{dyn}$, $X_i(t) \equiv X_i(t_0)$ (and we assume *uniform* initial composition)
- Hydrostatic equilibrium $(\partial^2 r/\partial t^2 = 0)$ + thermal equilibrium $(\partial s/\partial t = 0)$ = "Complete Equilibrium"
- Partial derivatives become ordinary derivatives and the quantities depend only on r
- Suitable boundary conditions (e.g. central and surface conditions)

Polytropic Models

- Mechanical equations are coupled to the "thermo-energetic" equations through the pressure $P = P(\rho, T)$
- For simple model $P = P(\rho)$ they decouple
- Polytropic Model $P = K\rho^\gamma$ with K *polytropic constant* and γ *polytropic exponent* (also defined as $\gamma = 1 + \frac{1}{n}$ with n *polytropic index*)
- K can be fixed from natural constants or can be a free parameter
- A simple case is the homogeneous gaseous sphere with $\rho = K' P^{1/\gamma}$ and $n = 0$ ($\gamma \to +\infty$)

Lane-Emden Equation

- By defining the dimensionless variables ξ and θ as $r = \alpha\xi$ and $\rho = \rho_c \theta^n$ we get the *Lane-Emden* equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0 \quad \text{with} \quad \alpha^2 = \frac{(n+1)K\rho_c^{-1+1/n}}{4\pi G}$$

- As boundary conditions, we require $\rho(0) = \rho_c$ and $\rho'(0) = 0$ in $r = 0$, i.e. $\theta(0) = 1$ and $\theta'(0) = 0$ in $\xi = 0$
- The radius is determined by $R = \alpha\xi_0$
- Actually, we could implicitly have the third condition $\rho(R) = 0$ if we assume $P(R) \approx 0$, i.e. $\theta(\xi_0) = 0$

General Properties of the Solutions

- Only for $n = 0, 1$ and 5 analytical solutions exist
- For $n < 5$ the solution admits *finite* radius and *finite* total mass
- For $n = 5$ the solution admits *infinite* radius but *finite* total mass[1]
- For $n \geq 5$ both quantities are *infinite*
- Quantitatively

$$m(\xi) = 4\pi\alpha^3\rho_c\left(-\xi^2\frac{d\theta}{d\xi}\right), \quad M \equiv m(\xi_0)$$

- Gravitational potential energy

$$E_g = -\frac{3}{5-n}\frac{GM^2}{R}$$

[1] The $n = 5$ solution is also known as Schuster-sphere. This case describes well the Lynds Dark Nebula https://arxiv.org/abs/astro-ph/0408089

Solutions for ρ/ρ_c for $n = 1, \frac{3}{2}$ and 3

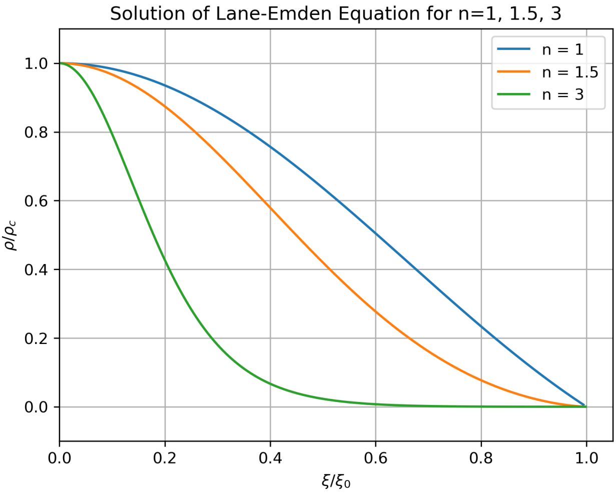

Figure 1: Plot of ρ/ρ_c as a function of ξ/ξ_0

Solutions for the ratio m/M for $n = 1, \frac{3}{2}$ and 3

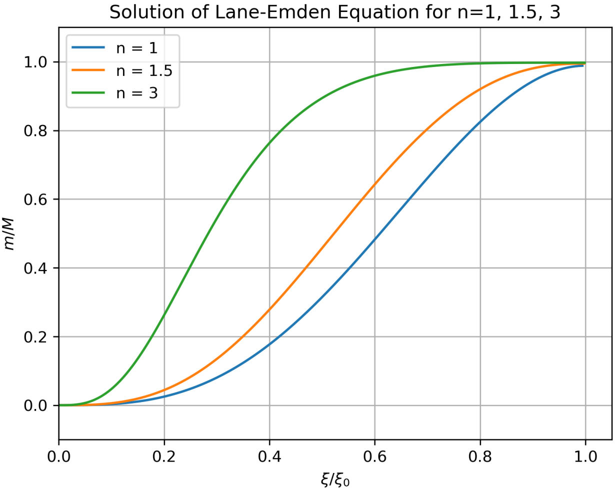

Figure 2: Plot of m/M as a function of ξ/ξ_0

Various Polytropes and the Sun

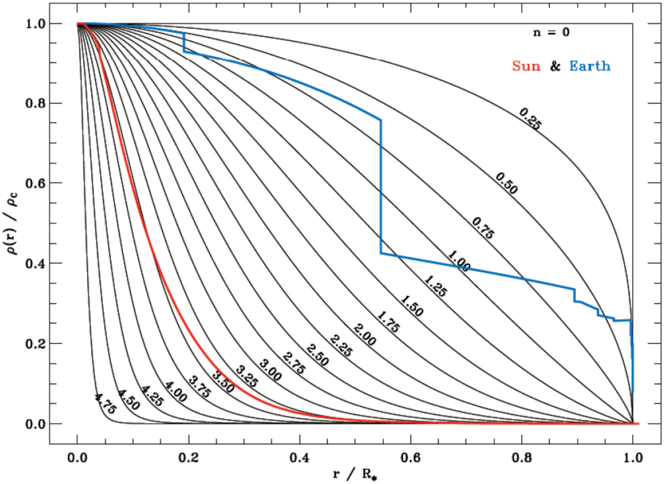

Figure 3: Various polytropes along with the Sun (red line). Source: https://www.ucolick.org/ woosley/ay112-14/lectures/lecture7.14.pdf

$n = 1, \frac{3}{2}$ and 3

- It can be demonstrated that physical solutions follow a mass-radius scaling of the form

$$M \sim R^{(n-3)/(n-1)}, \quad R \sim M^{(n-1)/(n-3)}$$

- For $n = 1$, radius is independent of the mass and dependent only on K
- For $n = 3/2$, we have $R \sim M^{-1/3}$ and more massive stars are more compact (and have higher density). This is the case of the *degenerate non-relativistic electron gas*, $P \propto \rho^{5/3}$
- For $n = 3$, mass is independent of the radius and given by

$$M_{Ch} = 5.836 \mu_e^{-2} M_\odot$$

where K is fixed from natural constants. This is the case of the *degenerate relativistic electron gas*, $P \propto \rho^{4/3}$

If K is a free parameter...(pt. 1)

- So far, we assumed K is determined by natural constants. Here we consider it as a free parameter. We assume the pressure is given by a mixture of ideal gas pressure and radiation pressure

$$P = P_{gas} + P_{rad} = \frac{R}{\mu}\rho T + \frac{a}{3}T^4$$

- We further assume the ratio $\beta = P_{gas}/P$, $1-\beta = P_{rad}/P$ is constant, i.e.

$$P = \frac{P_{rad}}{1-\beta} = \frac{aT^4}{3(1-\beta)} \Longrightarrow T = \left(\frac{3(1-\beta)}{a}\right)^{1/4} P^{1/4}$$

- Since $P = P_{gas}/\beta$, we have

$$P = \left(\frac{3R^4}{a\mu^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3} \equiv K\rho^{4/3}$$

which is a polytropic relation with $n = 3$, but K is a free parameter

If K is a free parameter...(pt. 2)

- We have already seen that for $n = 3$ the mass is dependent only of K, explicitly

$$M = 4\pi \left(-\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_0} \left(\frac{K}{\pi G}\right)^{3/2}$$

- By using the K as defined above, we have the "Eddington's quartic equation"

$$1 - \beta = 3.02 \cdot 10^{-3} \left(\frac{M}{M_\odot}\right)^2 \mu^4 \beta^4$$

- Since $M \to 0$ for $\beta \to 1$ and $M \to +\infty$ for $\beta \to 0$, we can conclude that the larger the mass, the more important the radiation pressure is
- For the Sun, $\mu_\odot \simeq 0.6$, so $\beta_\odot \simeq 0.99961$

If K is a free parameter...(pt. 3)

- By using the hydrostatic equilibrium and the transport equation, we can derive this mass-luminosity relation

$$L = 7.73 \left(\frac{\kappa_{s,\odot}}{\kappa_s} \right) \left(\frac{M}{M_\odot} \right)^3 \mu^4 \beta^4 L_\odot$$

  where κ_s is the opacity near the surface and $\kappa_{s,\odot} \simeq 1 cm^2 g^{-1}$
- For M close to or less than $M_\odot$, $\beta \simeq 1$, so $L \sim M^3$
- For higher masses, $\beta^4 \sim M^{-2}$ and $L \sim M$

The case $n = +\infty$ (pt. 1)

- This case corresponds to $\gamma = 1$, i.e. $P = K\rho$ (isothermal gas). Both mass and radius are infinite
- From hydrostatic equilibrium, $\rho(r) = \rho_c e^{-\Phi(r)/K}$, where $\Phi(r)$ satisfies the Poisson equation

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\rho(r)$$

- Again, we use dimensionless variables defined as $\xi = Ar$ and $\Phi = K\theta$ with $A^2 = 4\pi G\rho_c/K$ to get the *Emden–Chandrasekhar* equation

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} = e^{-\theta}$$

  with boundary conditions $\theta(0) = 0$ and $\theta'(0) = 0$
- Since $\xi = 0$ is a singular point, the power series expansion near this point gives $\theta(\xi) = \xi^2/6 - \xi^4/120$, hence $\theta''(0) = 1/3$

The case $n = +\infty$ (pt. 2)

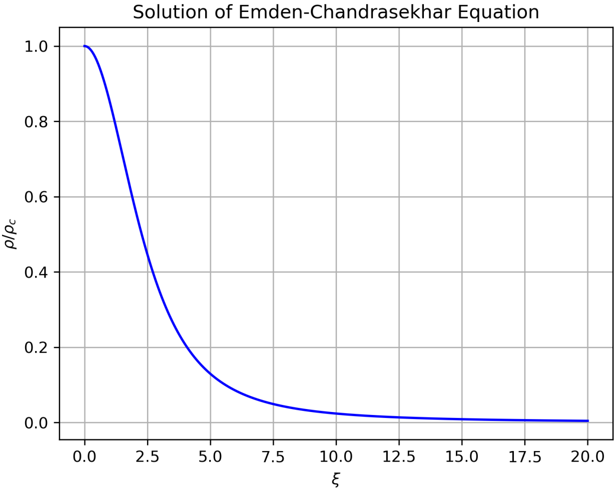

Figure 4: Plot of ρ/ρ_c as a function of ξ

The case $n = +\infty$ (pt. 3)

- The model is used to construct isothermal cores
- In this model, there exists a maximum isothermal mass known as *Schönberg–Chandrasekhar* limit and given by

$$q_{SC} = 0.37 \left(\frac{\mu_{env}}{\mu_{core}} \right)^2$$

  where $q_{SC} = M_{core}/M_{tot}$
- For example, when hydrogen burning stops, the core is mainly composed by helium, enveloped by hydrogen in outward shells

Thanks for your attention![2]

[2]Sources: 1) V. Castellani, Astrofisica stellare, Zanichelli (1985)
2) R. Kippenhahn A. Weigert, Stellar Structure and Evolution, Springer (1990)