Schönberg–Chandrasekhar limit

Stellar Astrophysics Course

Davide Serafini

Università degli Studi di Siena

October 24, 2023

Overview

Overview

1. Equations of State of the Stellar Matter
   - Introduction
   - Fully ionized perfect gas
2. Equations of Stellar Structure
3. The Hydrogen Burning Phase

Physical conditions of the stellar matter

Composition

- Stars are made of a mixture of gas plus radiation [1]

Thermodynamics

The thermodynamics properties of the stellar matter are described by the equation of state (EOS) which uniquely determines:

- the fractions of free electrons, neutral and ionized atoms
- their ionization states
- the thermodynamic quantities such as pressure P, temperature T, density ρ
- chemical composition of the gas

Thermodynamics

- From statistical mechanics the free energy of a gas is

$$F = -K_B T \ln \Xi \qquad \text{with } \Xi = \text{partition function} \tag{1}$$

Equilibrium Equations

- We get the internal energy per gram E

$$E = F - T \left(\frac{dF}{dT} \right)_{\rho,\mu} \tag{2}$$

- and the pressure

$$P = \rho^2 \left(\frac{dF}{d\rho} \right)_{T,\mu} \tag{3}$$

Fully ionized perfect gas

Perfect monoatomic gas

Two conditions defines a perfect gas:

- the potential energy of interaction between the gas particles is negligible with respect to their kinetic energy

$$\frac{Z^2 e^2}{d^2} < K_B T \tag{4}$$

- de Broglie wavelength associated to the gas particles is much smaller than their mean separation d

$$\lambda_{\mathrm{dB}} = \frac{h}{p} \ll d \tag{5}$$

where h is the Planck constant and p is the momentum of the particle

Mean distance of gas particles

1-dimensional

- one can think about a line of identically spaced points with distance d
- considering a line density of points n, there will be only one particle within length d

$$nd = 1 \tag{6}$$

3-dimensional

- considering a 3D space density of points n, there will be only one particle within a sphere of diameter d

$$\frac{4}{3}\pi\left(\frac{d}{2}\right)^3 n = 1 \implies d^3 = \frac{6}{n\pi} = \frac{6\mu m_H}{\pi\rho} \approx \frac{\mu m_H}{\rho} \implies d = \sqrt[3]{\frac{\mu m_H}{\rho}} \tag{7}$$

Mass of gas particles

Mass of gas particles

- consider a gas made of particles with average mass

$$< m >= \frac{\sum_{i=1}^{N} m_i}{N} = \frac{\rho \mathrm{Vol}}{n \mathrm{Vol}} = \frac{\rho}{n} \tag{8}$$

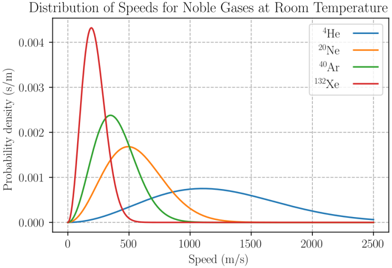
- the mean molecular weight is the average particle mass normalized to the atomic mass unit m_H

$$\mu = \frac{< m >}{m_H} = \frac{\rho}{n m_H} \qquad \text{with} \quad [\mu] = \mathrm{adim} \tag{9}$$

where ρ is the density and n the number per unit volume of the gas particles

Fully ionized perfect gas

- this implies that the momenta p and kinetic energies E_{kin} of the various particle species follow a Maxwell-Boltzmann distribution



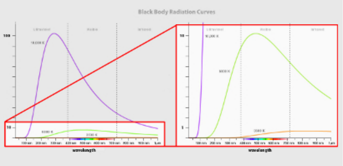
Maxwell-Boltzmann equations

$$n(p)dp = \sqrt{2}\pi N \left(\frac{1}{\pi m K_B T} \right)^{3/2} p^2 e^{-\frac{p^2}{2mK_B T}} dp \qquad (10)$$

$$n(E_{\mathrm{kin}})dE_{\mathrm{kin}} = 2\pi N \left(\frac{1}{\pi K_B T} \right)^{3/2} \sqrt{E_{\mathrm{kin}}}\, e^{-\frac{E_{\mathrm{kin}}}{K_B T}} dE_{\mathrm{kin}} \qquad (11)$$

where N denotes the total number of particles in the system and m their mass

Fully ionized perfect gas



Free energy

- Radiation in thermodynamical equilibrium with matter gives free energy

$$F_{\rm rad} = -\frac{4}{3}\frac{\sigma T^4}{c\rho} \equiv \frac{aT^4}{3\rho} \tag{12}$$

where σ is the Stefan-Boltzmann constant and a is the black-body constant

- Fully ionized species have no free energy from bound states

$$F_{\rm int} = 0 \tag{13}$$

Fully ionized perfect gas

Internal energy per unit mass

- Using Eq. 2

$$E = \overbrace{\frac{3}{2}\frac{1}{\mu m_H}K_B T}^{kinetic} + \frac{\overbrace{aT^4}^{radiation}}{\rho} \qquad \text{with} \quad [E] = \frac{\text{energy}}{\text{mass}} \tag{14}$$

Pressure per unit mass

- Using Eq. 3

$$P = \frac{K_B}{\mu m_H}\rho T + \frac{aT^4}{3} \tag{15}$$

Overview

Basic assumptions

The standard theory of stellar evolution is based on the following assumptions: [2]

Stars are spherically symmetric systems made of matter plus radiation

- The effects of rotation and magnetic fields are negligible;

The evolution of the physical and chemical quantities describing a star is slow

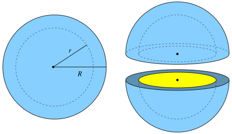
- The temporal evolution of the stellar structure can be described by a sequence of models in hydrostatic equilibrium;

Basic assumptions

The matter in each stellar layer is very close to local thermodynamic equilibrium

- the average distance travelled by particles between collisions is much smaller than the dimension of the system
- at each point within the star, radiation can be well described by the Planck function corresponding to the unique temperature in common with the matter
- each stellar layer can be assumed to behave like a black body

Continuity of mass



- Stars are spherically symmetric systems
- there is only 1 dof, the distance r from the centre

Calculating the continuity of mass equation

The mass contained within a sphere of radius r is

$$m_r = \int_0^r 4\pi r'^2 \rho dr' \tag{16}$$

Differentiating we get the continuity of mass equation

$$\frac{dr}{dm_r} = \frac{1}{4\pi r^2 \rho} \tag{17}$$

where ρ can be considered uniform over the infinitesimal shell element dm_r

Energy transport

Two ways to transport energy

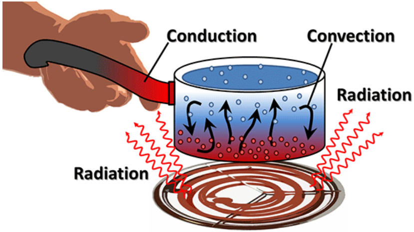
Inside a star energy can be transported either:

- by random motions of the constituent particles
- by organized large-scale motions of the matter

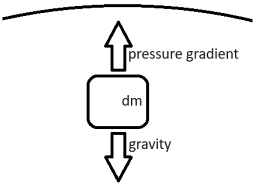
Three classes of transport mechanisms

Energy transport can be divided:

- radiative
- conductive
- convective



Hydrostatic equilibrium



Equation of motion for the volume element

The equation of motion of a generic infinitesimal cylindrical volume element with axis along the radial direction, located between radii r and $r + dr$ is

$$\frac{d^2 r}{dt^2} dm = -g(r) dm - \frac{dP}{dr} \frac{dm}{\rho} \tag{18}$$

where

$$g(r) = \frac{G m_r}{r^2} \tag{19}$$

is the inward gravitational acceleration

Hydrostatic equilibrium

Equation of hydrostatic equlibirum

The condition of hydrostatic equilibrium means that

$$\frac{d^2 r}{dt^2} = 0 \tag{20}$$

hence

$$\frac{dP}{dr} = -\frac{G m_r \rho}{r^2} \tag{21}$$

It can be rewritten with m_r as independent variable (Eq. 17)

$$\frac{dP}{dm_r} = -\frac{G m_r}{4\pi r^4} \tag{22}$$

Virial theorem

- The virial theorem plays a fundamental role in stellar evolution theory

Assumptions

- Consider a bound spherical gas system of mass M in hydrostatic equilibrium which can be described by Eq. 22
- assume that the temperatures are not high enough to start nuclear reactions

Virial theorem

Calculations

Multiplying both sides of Eq. 22 by $4\pi r^3$ and integrating over dm from the centre to the surface provides

$$\int_0^M \frac{dP}{dm} 4\pi r^3 dm = -\int_0^M \frac{Gm}{r} dm \quad (23)$$

An integration by part of the left-hand side of this equation gives

$$[4\pi r^3 p]_0^M - \int_0^M 12\pi r^2 \frac{dr}{dm} P dm = -\int_0^M \frac{Gm}{r} dm \quad (24)$$

Virial theorem

Calculations

Using the fact that $P \approx 0$ at the surface and $r = 0$ at the centre, and applying Eq. 17 to the second term on the left-hand side of this equation, we obtain

$$3 \int_0^M \frac{P}{\rho} dm = \int_0^M \frac{Gm}{r} dm \tag{25}$$

The right-hand side is $-\Omega$ where Ω denotes the total gravitational energy of the system. The left-hand side is related to the thermodynamics of the system.

Virial theorem

Assumptions

- assume a perfect monoatomic gas

Internal energy

Considering Eq. 14 and Eq. 15, the internal energy per unit mass can be expressed as

$$E = \frac{3}{2}\frac{P}{\rho} \tag{26}$$

then, the virial theorem can be represented as

$$E = -\frac{\Omega}{2} \tag{27}$$

Virial theorem

Total energy

The total energy E_t of the system is

$$E_t = E + \Omega \qquad \Longrightarrow \qquad E_t = -E = \frac{\Omega}{2} \tag{28}$$

Features of the described system

- the total energy is negative
- in agreement with the hypothesis that the system is bound

Virial theorem

Perturbation of hydrostatic equilibrium state

- Suppose a star with no active nuclear reactions
- Stellar surfaces are hotter than interstellar space
- Energy is radiated away (lost) from the surface

Hence the total energy E_t decreases according to

$$L = -\frac{dE_t}{dt} \quad = -\frac{1}{2}\frac{d\Omega}{dt} \tag{29}$$

Virial theorem

Gravitational contraction

- luminosity L must be positive
- so the star has to contract ($d\Omega < 0$)

Star heating

- internal energy has to increase correspondingly (Eq. 28)

$$\Delta E = -\Delta E_t = -\frac{\Delta\Omega}{2} \tag{30}$$

- So does the star temperature since $T \propto E$ (Eq. 14)

Virial theorem

Constraints on the average internal temperature of a star

- Suppose a star of mass M and radius R
- The gravitational energy is

$$\Omega = -\alpha \frac{GM^2}{R} \qquad \text{with} \quad \alpha(\text{density profile}) \sim 1 \stackrel{!}{=} \frac{3}{5} \tag{31}$$

- The internal energy for a monoatomic and fully ionized gas is

$$E \sim \frac{3}{2} K \bar{T} \frac{M}{\mu m_H} \tag{32}$$

where $\bar{T}$ is the mean temperature within the star

Virial theorem

Constraints on the average internal temperature of a star

- Using Eq. 28 and an average density $\bar{\rho} \propto M/R^3$ one finds

$$\bar{T} \propto M^{2/3}\bar{\rho}^{1/3} \tag{33}$$

Features of the average internal temperature

- to achieve the same mean temperature $\bar{T}$, less massive objects have to be denser

Virial theorem

Stability criterion for a star

- Consider a generic perfect gas
- the internal energy is

$$E = \frac{1}{\gamma - 1}\frac{P}{\rho} \qquad \text{with} \quad \gamma \equiv \frac{c_P}{c_v} \xrightarrow{\text{monoatomic}} \frac{5}{3} \tag{34}$$

- Substituting in Eq. 25 we get a more general expression for the virial theorem

$$E = -\frac{\Omega}{3(\gamma - 1)} \tag{35}$$

Virial theorem

Stability criterion for a star

- Correspondingly the total energy is

$$E_t = \frac{3\gamma - 4}{3(\gamma - 1)} \overbrace{\Omega}^{<0} \quad (36)$$

- a star will be in hydrostatic equilibrium ($E_t < 0$) only as long as

$$\gamma > \frac{4}{3} \quad (37)$$

Virial theorem

Non vanishing pressure at the surface

- Suppose pressure $P_0 > 0$ at the surface of the gas system
- the virial theorem becomes

$$3(\gamma - 1)E + \Omega = 4\pi R^3 P_0 \tag{38}$$

Overview

Overview

Regarding the H-burning phase

- it is the longest evolutionary phase
- the structural and evolutionary properties of a star during the central and shell H-burning phases determine its evolutionary properties through all successive phases
- its termination is related to the most important astrophysical clock
- the final portion of the shell H-burning phase in low-mass, metal-poor stars provides an accurate distance indicator for old stellar populations
- offers an opportunity for deriving the Inital Mass Function (IMF) of stellar systems

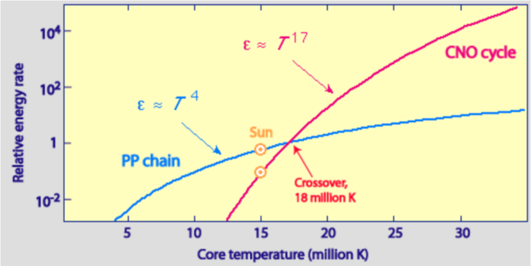
We define a star as being on the Main Sequence if its evolutionary rate is controlled by the timescale of the H-burning process occurring in the core

The nuclear reactions

H-burning mechanism

- The H-burning mechanism is essentially the nuclear fusion of four protons into one He-4 nucleus
- the nuclear conversion of H into He is very efficient
- the fusion of H nuclei can be achieved through two reactions chains, namely the p-p chain and the CNO cycle

The central H-burning phase in lower main sequence (LMS) stars

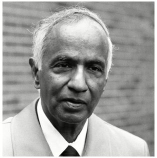


- In low-mass stars ($M \leq 1.3 M_{\odot}$) the main H-burning mechanism is the p-p chain
- In general, the star regulates its thermonuclear burning rate so that the nuclear energy production is just enough to enforce the hydrostatic equilibrium condition
- If the number of nuclear reactions is larger than needed, the star reacts by expanding, thus decreasing the temperature and density, so that the burning rate decreases and equilibrium is re-established

The central H-burning phase in upper main sequence (UMS) stars

- the main effect of an increase in the stellar mass is a significant increase in the interior temperature
- the most important effect of this temperature increase is that the CNO cycle becomes the dominant energy production mechanism

The Schönberg–Chandrasekhar limit



Chandrasekhar

Schönberg

- In 1942, Schönberg and Chandrasekhar investigated, with a pure analytical approach, the hydrostatic equilibrium conditions for an isothermal He core with an ideal gas EOS
- They found a fixed limiting value for the ratio between the core mass and the total stellar mass: $M_{\rm core}/M_{\rm tot}$
- Beyond the Schönberg–Chandrasekhar limit, the core must contract on Kelvin-Helmholtz timescales due to the envelope pressure

The Schönberg–Chandrasekhar limit

Assumptions

- at the exhaustion of central H the star is left with an He core surrounded by an H rich envelope [3]
- Suppose no nuclear burning inside the He core
- Suppose the temperature gradient is radiative
- Suppose its thermal stratification is isothermal

Then Eq. 75 can be used

$$\frac{dT}{dm_r} = -\frac{3k}{64\pi^2 ac}\frac{L_r}{r^4 T^3}$$

The Schönberg–Chandrasekhar limit

Equation

- Consider an isothermal core of radius R_c and mass $M_{\rm core}$
- Suppose a non-vanishing pressure P_0 at the surface of the core
- apply the virial theorem represented by Eq. 38
- Substitute for the internal energy and the gravitational energy respectively

$$E = K_1 M_{\rm core} T_c \qquad \text{and} \qquad \Omega = -K_2 \frac{M_{\rm core}^2}{R_c} \tag{39}$$

with $K_{1,2}$ constants. In these terms, the virial theorem becomes

$$K_1 M_{\rm core} T_c - K_2 \frac{M_{\rm core}^2}{R_c} = P_0 R_c^3 \tag{40}$$

The Schönberg–Chandrasekhar limit

Solution

- solving for P_0

$$P_0 = K_1 \frac{M_{\rm core} T_c}{R_c^3} - K_2 \frac{M_{\rm core}^2}{R_c^4} \tag{41}$$

- the first term in the right-hand side comes from the total internal energy of the core
- the second term comes from the gravitational potential Ω

Maximum

- We look for the core radius at which the surface pressure is on a maximum

$$\frac{dP_0}{dR_c} \overset{!}{=} 0 \tag{42}$$

The Schönberg–Chandrasekhar limit

Maximum

- We find that the surface pressure is maximum for

$$R_c = K_4 \frac{M_{\mathrm{core}}}{T_c} \tag{43}$$

- At this core radius, the surface pressure is

$$P_{0,\mathrm{m}} = K_3 \frac{T_c^4}{M_{\mathrm{core}}^2} \tag{44}$$

- the maximum pressure decreases for increasing M_{core}

The Schönberg–Chandrasekhar limit

Equilibrium

- For the star to be in equilibrium it is necessary to have

$$P_{0,\mathrm{m}} \geq P_e \tag{45}$$

where P_e is the pressure exerted by the non-degenerate envelope on the interface with the core

The Schönberg–Chandrasekhar limit

Envelope pressure at the interface

- apply the dimensional analysis to Eq. 22 and consider $\rho \propto M/R^3$
- assume that the functional dependencies found for the central values also hold for any other point within the star
- we can approximate

$$P_e \propto \frac{M_{\rm tot}^2}{R^4} \qquad \text{and} \qquad T_c \propto \frac{M_{\rm tot}}{R} \tag{46}$$

  where $M_{\rm tot}$ is the total mass of the star and R its radius
- hence the pressure exerted by the envelope is

$$P_e \propto \frac{T_c^4}{M_{\rm tot}^2} \tag{47}$$

The Schönberg–Chandrasekhar limit

Upper limit to the mass ratio

At the interface

- core pressure is $P_{0,\mathrm{m}} \propto M_{\mathrm{core}}^{-2}$
- envelope pressure is $P_e \propto M_{\mathrm{tot}}^{-2}$
- therefore the condition $P_{0,\mathrm{m}} \geq P_e$ becomes somewhat

$$M_{\mathrm{core}} \leq \mathrm{factor} \cdot M_{\mathrm{tot}} \tag{48}$$

- there exist an upper limit to the ratio $M_{\mathrm{core}}/M_{\mathrm{tot}}$
- beyond the Schönberg–Chandrasekhar limit the core must contract

The Schönberg–Chandrasekhar limit

Upper limit to the mass ratio

- the exact value of the mass ratio upper limit is

$$\left(\frac{M_{\rm core}}{M_{\rm tot}}\right)_{\rm SC} = 0.37\left(\frac{\mu_{\rm env}}{\mu_{\rm core}}\right)^2 \tag{49}$$

- Suppose a solar chemical composition with $\mu_{\rm env} \sim 0.6$ and $\mu_{\rm core} \sim 1.3$
- At the end of the MS phase the Schönberg–Chandrasekhar limit will be ~ 0.08
- if the He core mass is larger than 10 per cent of the total mass, it must contract

Intermediate-mass and massive stars

- $M_{\mathrm{core}} >$ S-C limit
- core contracts and envelope expands
- In the HRD the star moves from blue to red at almost constant luminosity (Sub-Giant Branch)
- Evolutionary rate is the Kelvin-Helmholtz timescale
- the envelope expands at constant effective temperature increasing luminosity (Red Giant Branch)

High-intermediate mass

- no electron degeneracy at the core
- core reaches $T \approx 10^8$K $\implies$ He-burning
- higher total mass $\implies$ shorter lifetime

Low-mass stars

- $M_{\rm core}$ < S-C limit
- electron gas in the He core becomes electron degenerate
- electron degeneracy of the He core pushes the envelope even beyond Schönberg–Chandrasekhar limit

Thank you for the attention

References I

[1] "Equation of State of the Stellar Matter". In: *Evolution of Stars and Stellar Populations*. John Wiley & Sons, Ltd, 2005. Chap. 2, pp. 31–47. ISBN: 9780470033456. DOI: https://doi.org/10.1002/0470033452.ch2. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/0470033452.ch2. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/0470033452.ch2.

[2] "Equations of Stellar Structure". In: *Evolution of Stars and Stellar Populations*. John Wiley & Sons, Ltd, 2005. Chap. 3, pp. 49–103. ISBN: 9780470033456. DOI: https://doi.org/10.1002/0470033452.ch3. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/0470033452.ch3. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/0470033452.ch3.

References II

[3] "The Hydrogen Burning Phase". In: *Evolution of Stars and Stellar Populations*. John Wiley & Sons, Ltd, 2005. Chap. 5, pp. 117–159. ISBN: 9780470033456. DOI: https://doi.org/10.1002/0470033452.ch5. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/0470033452.ch5. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/0470033452.ch5.

Overview

4 Back-up

- Time scales
- Electron degeneracy
- The Advanced Evolutionary Phases
- Conservation of energy
- Radiative Transport
- Conductive transport
- Overall picture of stellar evolution
- The nuclear reactions
- Overall picture of stellar evolution

Back-up

Nuclear time scale

- lifetime of a star based solely on its rate of fuel consumption via H-burning, $(4m_H - m_{\mathrm{He}})/m_H \sim 0.007$
- In reality, the lifespan of a star is greater than what is estimated by the nuclear time scale because as one fuel becomes scarce, another will generally take its place
- However, all the phases after hydrogen burning combined typically add up to less than 10 % of the duration of hydrogen burning
- the nuclear time scale is

$$\tau_n = \frac{E_n}{L} F \tag{50}$$

  where E_n is the nuclear energy reservoir, L is the luminosity and F the fraction of reservoir available
- assume that the sun is made of pure H and only the central 10 per cent of its mass is hot enough to undergo nuclear reactions, then $\tau_n \approx 1 \times 10^{10}$ y

Kelvin–Helmholtz time scale

- time it takes for a star to radiate away its total kinetic energy content at its current luminosity rate

$$\tau_{\mathrm{KH}} \approx \frac{E}{L} \tag{51}$$

- Using virial theorem the internal energy is about half the opposite of the gravitational energy, so

$$\tau = \frac{GM^2}{DL} \tag{52}$$

  where M and D are total mass and diameter of the star respectively

- for the sun $\tau_{\mathrm{KH}} \sim 2 \times 10^7$ y

Free-fall time scale

- time for the star collapse when the gravitational force is not balanced by the pressure
- consider the hydrostatic equation (Eq. 18) without the pressure gradient

$$\frac{d^2 r}{dt^2} dm = -G \frac{m_r}{r^2} dm \tag{53}$$

- Substitute the characteristic quantities for each dimension

$$\frac{R}{\tau_{\mathrm{ff}}^2} = G \frac{M}{R^2} \qquad \Longrightarrow \qquad \tau_{\mathrm{ff}} \sim \frac{R^{3/2}}{(GM)^{1/2}} \sim \sqrt{\frac{1}{G\bar{\rho}}} \tag{54}$$

- for the sun $\bar{\rho}_{\odot} \sim 1.4\,\mathrm{g/cc}$ and $\tau_{\mathrm{ff},\odot} \sim 30\,\mathrm{min}$

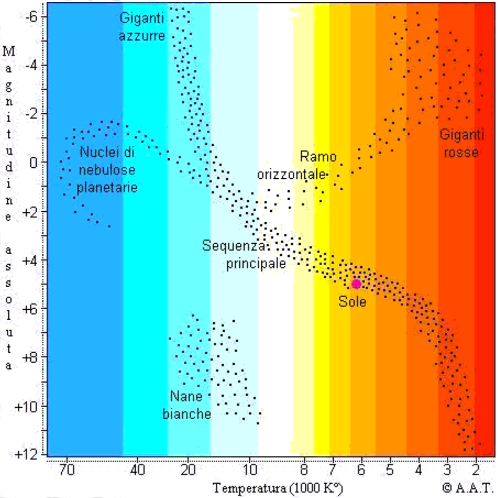
Time scales

Time scales comparison

In general

$$\tau_n \gg \tau_{\mathrm{KH}} \gg \tau_{\mathrm{ff}} \tag{55}$$

Hertzsprung-Russell diagram



Electron degeneracy

- the interparticle distance is comparable to the de Broglie wavelength
- temperature low enough at a given density
- Maxwell distribution model is not valid anymore
- Fermi-Dirac distribution must be used

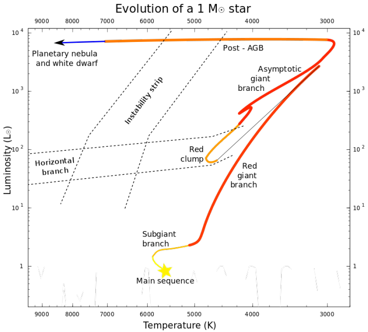
$$n(p)dp = \frac{8\pi p^2}{h^3}\left(\frac{1}{1+e^{-\eta+E\mathrm{kin}/K_B T}}\right)dp \tag{56}$$

- in the case of electron degeneracy the electron pressure does not depend on the temperature

$$P = 1.0036 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \tag{57}$$

The asymptotic giant branch (AGB)

- The abundance of He becomes low enough
- the stellar tracks move on the Hertzsprung–Russell diagram (HRD) towards a lower effective temperature and larger luminosity



- It corresponds to the He-shell burning phase
- In low-mass AGB stars the effective temperature-luminosity relationship is very similar to that of low-mass RGB stars

Conservation of energy

Equation of energy conservation

If some energy is produced in a spherical shell of thickness dr, located at distance r from the centre of the star, the local luminosity is equal to

$$dL_r = 4\pi r^2 \rho dr \epsilon \tag{58}$$

where ϵ denotes the coefficient of energy generation per unit time and unit mass. Then the equation of energy conservation is

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon \tag{59}$$

It can be expressed in term of m_r as

$$\frac{dL_r}{dm_r} = \epsilon \tag{60}$$

Radiative transport I

Assumptions

- Consider a net flux of photons crossing a volume element of unit area and depth dr located at distance r from the centre of the star
- While crossing some photons will be extracted from the net outgoing flux due to interactions with neighbouring particles
- these photons will be redistributed isotropically
- assume the properties of these photons do not change along their mean free path

Radiative transport II

Transferred momentum

The momentum dp transferred from the photons to the volume element is equal to

$$dp = \frac{dF_{\mathrm{rad}}}{c} = \frac{F_{\mathrm{rad}}}{c}\frac{dr}{l} \tag{61}$$

where F_{rad} is the flux of energy of the outgoing photons and l is the photon mean free path. On the other hand dp is also equal to the opposite of the change dP_{rad} of the pressure exerted by the photons over the length dr

$$dp = -dP_{\mathrm{rad}} \tag{62}$$

Therefore

$$dP_{\mathrm{rad}} = -\frac{F_{\mathrm{rad}}}{c}\frac{dr}{l} \tag{63}$$

Radiative transport III

Interaction probability

We introduce the opacity coefficient k_{rad} as

$$k_{\mathrm{rad}}\rho = \frac{1}{l} \tag{64}$$

which is a measure of the probability that the photons experience one interaction per unit length. Hence

$$\frac{dP_{\mathrm{rad}}}{dr} = -\frac{k_{\mathrm{rad}}\rho}{c}F_{\mathrm{rad}} \tag{65}$$

Radiative transport IV

Total energy flux

The black body assumption provides in Eq. 15

$$P_{\rm rad} = a\frac{T^4}{3} \qquad \text{with derivative} \qquad \frac{dP_{\rm rad}}{dr} = \frac{4}{3}aT^3\frac{dT}{dr} \tag{66}$$

We can rewrite Eq. 65 and obtain the equation of radiative transport in stellar interiors, when energy is carried by photons

$$\frac{dT}{dr} = -\frac{3k_{\rm rad}\rho}{4acT^3}F_{\rm rad} \tag{67}$$

If the total energy flux is carried by photons then we can express the flux of energy $F_{\rm rad}$ using the luminosity $L_r = F_{\rm rad}4\pi r^2$

$$\frac{dT}{dr} = \frac{3k_{\rm rad}\rho}{4acT^3}\frac{L_r}{4\pi r^2} \tag{68}$$

Conductive transport I

Energy transport due to free non-degenerate electrons

- the energy is transported by the constituents of the stellar matter other than the photons, i.e. free non-degenerate electrons

The energy flux is given approximately by

$$F_e \sim -N_e \nu l \frac{dE}{dr} \tag{69}$$

where N_e is the number of electrons per unit volume, ν their average velocity, l the mean free path and E their average kinetic energy. Since $E \propto K_B T$,

$$F_e \sim -K_B N_e \nu l \frac{dT}{dr} \tag{70}$$

Conductive transport II

Analogy with radiative transport

Note that Eq. 70 has the same form as Eq. 67. Analogously, we introduce the electron opacity k_e

$$\frac{dT}{dr} \sim -\frac{F_e}{K_B N_e \nu l} \tag{71}$$

Features of conductive transport

- electron transport is very inefficient with respect to radiation
- ions are outnumbered by electrons
- ion transport is even more inefficient
- degenerate electrons transport energy much more efficiently

Conductive transport III

Radiative plus conductive transport

Therefore the total energy flux will be

$$F = F_{\mathrm{rad}} + F_e \tag{72}$$

The equation of the radiative plus conductive energy transport for the stellar interiors becomes

$$\frac{dT}{dr} = -\frac{3k\rho}{4acT^3}\frac{L_r}{4\pi r^2} \tag{73}$$

where k is the total opacity of the stellar matter given by

$$\frac{1}{k} = \frac{1}{k_{\mathrm{rad}}} + \frac{1}{k_e} \tag{74}$$

If electron conduction is effective, $k_e \ll k_{\mathrm{rad}}$ and therefore $k \sim k_e$.

Conductive transport IV

Radiative plus conductive transport

We can rewrite the equation for the radiative plus energy transport in terms of m_r as

$$\frac{dT}{dm_r} = -\frac{3k}{64\pi^2 ac}\frac{L_r}{r^4 T^3} \quad (75)$$

The only mechanism of chemical element transport within stars is convection

- the effect of rotational mixing and atomic diffusion is negligible

Overall picture of stellar evolution

- the higher the mass the shorter the lifetime

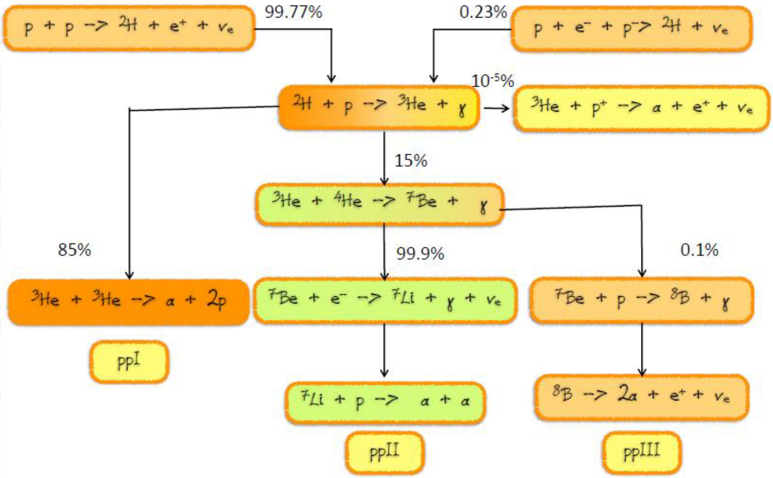
$$\uparrow M \implies \downarrow \tau_{\mathrm{life}} \tag{76}$$

- the lower the mass the higher the central density and the lower the central temperature

$$\downarrow M \implies \uparrow \rho_c \downarrow T_c \tag{77}$$

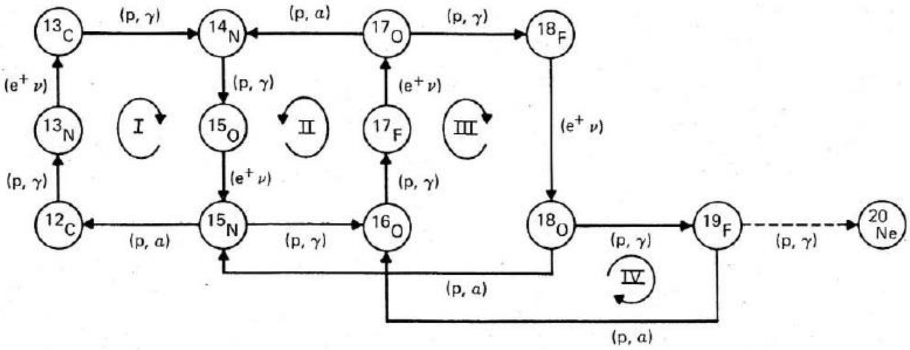
- the higher the metallicity the lower the luminosity and the T_{eff}
- the higher the initial He content the higher the luminosity and the T_{eff} and the shorter the evolutionary time scale

The p-p chain



p + p -> 2H + e+ + νe
99.77%
0.23%
p + e- + p -> 2H + νe
2H + p -> 3He + γ
10-5%
3He + p+ -> α + e+ + νe
15%
3He + 4He -> 7Be + γ
85%
99.9%
0.1%
3He + 3He -> α + 2p
7Be + e- -> 7Li + γ + νe
7Be + p -> 8B + γ
ppI
7Li + p -> α + α
8B -> 2α + e+ + νe
ppII
ppIII

The CNO cycle



The dependence of the MS tracks on chemical composition and convection efficiency

How stellar evolution models depend on the adopted inputs:

- the inital abundance of helium affects the MS evolution through the changes induced in the radiative opacity and the mean molecular weight
- a change in metallicity affects the radiative opacity much more than the nuclear energy generation
- the efficiency of convection

The mass-luminosity relation

- Stellar evolution models predict that stars spend a sizeable fraction of their core H-burning evolutionary lifetime close to their ZAMS location
- Stars on or near the ZAMS display a tight relationship between their total mass and the surface luminosity

Overall picture of stellar evolution

- the higher the mass the shorter the lifetime

$$\uparrow M \implies \downarrow \tau_{\rm life} \tag{78}$$

- the lower the mass the higher the central density and the lower the central temperature

$$\downarrow M \implies \uparrow \rho_c \downarrow T_c \tag{79}$$

The stellar mass has a pivotal role in stellar evolution

Very low-mass stars

- the evolutionary and structural properties of low-mass stars change significantly when entering into the realm of very low-mass (VLM) stars
- stars below $0.3M_{\odot}$ are fully convective all along their MS lifetime